Numerische Mathematik, 1./2. Vordiplom Herbst 2002

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1

1.1

$$\begin{aligned}
det(A) - \underbrace{\lambda}_{=1} E &= \begin{vmatrix} 0 & -1 & 2 \\ -1 & 2 & -4 \\ 2 & -4 & a-1 \end{vmatrix} \\
&= (-1)(-4)2 + 2(-1)(-4) - (-1)(-1)(a-1) - (2\,2\,2) \\
&= 9 - a \overset{!}{=} 0 \Rightarrow a = 9 \\
\Rightarrow A &= \begin{pmatrix} 1 & -1 & 2 \\ -1 & 3 & -4 \\ 2 & -4 & 9 \end{pmatrix}
\end{aligned}$$

1.2

$$\begin{aligned}
cond_x(A) &= \|A^{-1}\|_x \|A\|_x = \kappa \\
\|A\|_2 &= \sqrt{\max|EW von A^T A|}
\end{aligned}$$

Da A symmetrisch ist $\lambda_A^2 = \lambda_{A^TA}$ (), wir suchen also $\max|A|$ und $\max|A^{-1}|$

$$\begin{aligned}
\chi_A &= (1-\lambda)(3-\lambda)(9-\lambda) + 8 + 8 - (1-\lambda)16 - (9-\lambda) - 2(3-\lambda)2 \\
&= -\lambda^3 + 13\lambda^2 - 18\lambda + 6
\end{aligned}$$

Da ein $EW = 1$ Polynomdivision mit $(\lambda - 1)$

$$\begin{aligned}
&\Rightarrow (\lambda-1)(-\lambda^2 + 12\lambda - 6) \\
&\Rightarrow \lambda_{A_1} = 1 \\
&\quad \lambda_{A_2} = \frac{-12 + \sqrt{120}}{-2} = 6 + \sqrt{30} \\
&\quad \lambda_{A_3} = \frac{-12 - \sqrt{120}}{-2} = 6 - \sqrt{30} \\
&\Rightarrow \max|EW| = 6 + \sqrt{30}
\end{aligned}$$

Da $AA^{-1} = E$ ist auch $diag(A)diag(A^{-1}) = E$ und daher gilt $\frac{1}{\lambda_A} = \lambda_{A^{-1}}$

$$\begin{aligned}
\Rightarrow \lambda_{A_1^{-1}} &= 1 \\
\lambda_{A_2^{-1}} &= \frac{1}{6+\sqrt{30}} \\
\lambda_{A_3^{-1}} &= \frac{1}{6-\sqrt{30}} \\
\Rightarrow \max|EW| &= \frac{1}{6-\sqrt{30}}
\end{aligned}$$

$$\begin{aligned}
cond_2(A) &= \left(6+\sqrt{30}\right)\frac{1}{6-\sqrt{30}} = \frac{\left(6+\sqrt{30}\right)^2}{\left(6-\sqrt{30}\right)\left(6+\sqrt{30}\right)} \\
&= 11 + 2\sqrt{30} \approx 21.95
\end{aligned}$$

1.3

$$R = \begin{pmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{pmatrix} = \begin{pmatrix} \sqrt{a_{11}} & \frac{a_{12}}{r_{11}} & \frac{a_{13}}{r_{11}} \\ 0 & \sqrt{a_{22} - r_{12}^2} & \frac{a_{23} - r_{12}r_{13}}{r_{22}} \\ 0 & 0 & \sqrt{a_{33} - r_{13}^2 - r_{23}^2} \end{pmatrix}$$

$$\begin{pmatrix} \sqrt{1} & \frac{-1}{1} & \frac{2}{1} \\ 0 & \sqrt{3-(-1)^2} & \frac{-4-(-1)2}{\sqrt{2}} \\ 0 & 0 & \sqrt{9 - 2^2 - (-\sqrt{2})^2} \end{pmatrix} = \begin{pmatrix} 1 & -1 & 2 \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & \sqrt{3} \end{pmatrix}$$

1.4

$$\begin{aligned}
cond_2(R) &= \sqrt{\max|EW von (R^{-1})^T R^{-1}|}\sqrt{\max|EW von R^T R|} \\
&= \sqrt{\max|EW von A^{-1}|}\sqrt{\max|EW von A|} \\
&\overset{\star}{=} \sqrt{\sqrt{\max|EW von (A^{-1})^T A^{-1}|}}\sqrt{\sqrt{\max|EW von A^T A|}} \\
&= \sqrt{\sqrt{\max|EW von (A^{-1})^T A^{-1}|}\sqrt{\max|EW von A^T A|}} \\
&= \sqrt{cond_2(A)}
\end{aligned}$$

Der Schritt bei $\star$ folgt aus 1.2 da eine Cholesky-Zerlegung nur möglich ist, wenn die Matrix symmetrisch positiv definit ist.

2

2.1

i	x
0	-4
1	-1
2	4

Inverseinterpolation

i	y	f_i	$f[y_i, y_{i+1}]$	$f[y_i, y_{i+1}, y_{i+2}]$
0	-4	1		
1	-1	2	$\frac{1}{3}$	
2	4	3	$\frac{1}{5}$	$-\frac{1}{300}$

Newton Polynom

$$1 + \frac{1}{3}(x+4) - \frac{5}{300}(x+4)(x+1) \text{ ausgewärtet an der Stelle } x = 0 \ \Rightarrow \frac{34}{15}$$

2.2

$$\frac{1}{n+1!} \max_{x \in [-4,4]} |f^{(n+1)}(x)||\Pi_{i=0}^{n}(x - x_i)|$$

3

3.1

Fehler ϵ von: $T(n)$ wenn $I = \frac{2}{\sqrt{\pi}} \int f(x)\, dx$: mit $f(x) = e^{-x^2}$

$$\epsilon = |I - T(n)| \leq \frac{h^2}{12}(b-a)\frac{2}{\sqrt{\pi}} \max_{x \in [a,b]} |f''(x)|$$

$$\text{Anzahl Stützstellen} = n$$

$$h = \frac{b-a}{n}$$

$$\text{gesucht: } g(x, \epsilon) = n$$

$$f'(x) = -2xe^{-x^2} \Rightarrow f''(x) = (4x^2 - 2)e^{-x^2}$$

$$\epsilon \leq \frac{\left(\frac{x-0}{n}\right)^2}{12}(x-0)\frac{2}{\sqrt{\pi}} \max_{x \in [a,b]} |(4x^2 - 2)e^{-x^2}|$$

$$\max_{x\in[a,b]}|(4x^2-2)e^{-x^2}| \Rightarrow \left((4x^2-2)e^{-x^2}\right)' = 8xe^{-x^2} - (4x^2-2)2xe^{-x^2}$$

$$= (-4x^2+4x+2)2xe^{-x^2} \stackrel{!}{=} 0$$

$$\Rightarrow (-2x^2+2x+1)xe^{-x^2} \stackrel{!}{=} 0$$

$$\Rightarrow x = 0 \Rightarrow x = 0$$

$$\Rightarrow e^{-x^2} = 0 \Rightarrow x = \infty$$

$$\Rightarrow -2x^2+2x+1 = 0 \Rightarrow x = \frac{-2\pm\sqrt{12}}{-4} = \frac{1}{2} \pm \frac{\sqrt{12}}{4}$$

Einsetzen der Resultate in $\max_{x\in[a,b]}|f(x)''|$ ergibt $x = 0$ als Lösung und $|f(0)''| = 2$.

$$\epsilon \leq \frac{\left(\frac{x-0}{n}\right)^2}{12}(x-0)\frac{2}{\sqrt{\pi}}2 = \frac{x^3}{3\sqrt{\pi}n^2} \Rightarrow n \geq \sqrt{\frac{x^3}{3\sqrt{\pi}\epsilon}}$$

3.2

$$n \geq \sqrt{\frac{x^3}{3\sqrt{\pi}\epsilon}} = \sqrt{\frac{1^3}{3\sqrt{\pi}0.02}} \approx 3.07 \Rightarrow n = 4$$

$$I = \frac{2}{\sqrt{\pi}}\int f(x)\,dx \text{ mit } f(x) = e^{-x^2}$$

$$f(0) = 1$$

$$f\left(\frac{1}{4}\right) = e^{-\frac{1}{16}} \approx 0.939$$

$$f\left(\frac{1}{2}\right) = e^{-\frac{1}{4}} \approx 0.779$$

$$f\left(\frac{3}{4}\right) = e^{-\frac{9}{16}} \approx 0.570$$

$$f(1) = e^{-1} \approx 0.368$$

$$T_{normal}(n) = \frac{b-a}{n}\left(\frac{1}{2}f(x_0) + \sum_{i=1}^{n-1} f(x_i) + \frac{1}{2}f(x_n)\right) \text{ ohne } \frac{2}{\sqrt{\pi}}$$

$$\Rightarrow \frac{2}{\sqrt{\pi}}\frac{1}{4}\left(\frac{1}{2}1 + e^{-\frac{1}{16}} + e^{-\frac{1}{4}} + e^{-\frac{9}{16}} + \frac{1}{2}e^{-1}\right) \approx 0.838 \pm 0.02$$

4

4.1

Periodischer Spline $M_0 = M_n$

$$\begin{pmatrix} 2 & \lambda_1 & 0 & \cdots & 0 & \mu_1 \\ \mu_2 & 2 & \ddots & & & 0 \\ 0 & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & 0 \\ 0 & & & \ddots & 2 & \lambda_{n-1} \\ \lambda_n & 0 & \ddots & 0 & \mu_n & 2 \end{pmatrix} \begin{pmatrix} M_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ M_n \end{pmatrix} = \begin{pmatrix} d_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ d_n \end{pmatrix}$$

$$\mu_i = \frac{h_i}{h_i + h_{i+1}} = \frac{1}{2} \qquad \lambda_i = 1 - \mu_i = \frac{1}{2}$$

$$d_i = \frac{6}{h_i} f[x_i, x_{i+1}, x_{i+2}]) \Rightarrow \vec{d} = \begin{pmatrix} -9 \\ -15 \\ 3 \\ -21 \end{pmatrix}$$

Schema der dividierten Differenzen

$$\begin{array}{cccccc} x_i & f_i & & f[x_i, x_{i+1}] & & f[x_i, x_{i+1}, x_{i+2}] \\ 0 & -2 & & & & \\ 1 & \frac{5}{2} & \searrow\rightarrow & \frac{9}{2} & & \\ 2 & 4 & \searrow\rightarrow & \frac{3}{2} & \searrow\rightarrow & -\frac{3}{2} \\ 3 & \frac{1}{2} & \searrow\rightarrow & -\frac{7}{2} & \searrow\rightarrow & -\frac{5}{2} \\ 4 & -2 & \searrow\rightarrow & -\frac{5}{2} & \searrow\rightarrow & \frac{1}{2} \\ 5 & \frac{5}{2} & \searrow\rightarrow & \frac{9}{2} & \searrow\rightarrow & \frac{7}{2} \\ 6 & 4 & \searrow\rightarrow & \frac{3}{2} & \searrow\rightarrow & -\frac{3}{2} \\ \vdots & \vdots & & \vdots & & \vdots \end{array}$$

Momentengleichung

$$\begin{pmatrix} 2 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 2 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 2 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix} = \begin{pmatrix} -9 \\ -15 \\ 3 \\ -21 \end{pmatrix}$$

Gauss-Algorithmus

$$\begin{pmatrix} 2 & \frac{1}{2} & 0 & \frac{1}{2} \\ \frac{1}{2} & 2 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{2} & 0 & \frac{1}{2} & 2 \end{pmatrix} \Rightarrow \begin{array}{c|ccc} 2 & \frac{1}{2} & 0 & \frac{1}{2} \\ \hline \frac{1}{4} & \frac{15}{8} & \frac{1}{2} & -\frac{1}{8} \\ 0 & \frac{1}{2} & 2 & \frac{1}{2} \\ \frac{1}{4} & -\frac{1}{8} & \frac{1}{2} & \frac{15}{8} \end{array} \Rightarrow$$

$$\begin{array}{c|c|cc} 2 & \frac{1}{2} & 0 & \frac{1}{2} \\ \hline \frac{1}{4} & \frac{15}{8} & \frac{1}{2} & -\frac{1}{8} \\ \cline{2-4} 0 & \frac{4}{15} & \frac{28}{15} & \frac{8}{15} \\ \frac{1}{4} & -\frac{1}{15} & \frac{8}{15} & \frac{28}{15} \end{array} \Rightarrow \begin{array}{c|c|c|c} 2 & \frac{1}{2} & 0 & \frac{1}{2} \\ \hline \frac{1}{4} & \frac{15}{8} & \frac{1}{2} & -\frac{1}{8} \\ \cline{2-4} 0 & \frac{4}{15} & \frac{28}{15} & \frac{8}{15} \\ \cline{3-4} \frac{1}{4} & -\frac{1}{15} & \frac{2}{7} & \frac{12}{7} \end{array}$$

Vorwärtseinsetzen

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{4} & 1 & 0 & 0 \\ 0 & \frac{4}{15} & 1 & 0 \\ \frac{1}{4} & -\frac{1}{15} & \frac{2}{7} & 1 \end{pmatrix} \vec{y} = \begin{pmatrix} -9 \\ -15 \\ 3 \\ -21 \end{pmatrix} \Rightarrow \vec{y} = \begin{pmatrix} -9 \\ -\frac{51}{4} \\ \frac{32}{5} \\ -\frac{148}{7} \end{pmatrix}$$

Rückwärtseinsetzen

$$\begin{pmatrix} 2 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{15}{8} & \frac{1}{2} & -\frac{1}{8} \\ 0 & 0 & \frac{28}{15} & \frac{8}{15} \\ 0 & 0 & 0 & \frac{12}{7} \end{pmatrix} \vec{M} = \begin{pmatrix} -9 \\ -\frac{51}{4} \\ \frac{32}{5} \\ -\frac{148}{7} \end{pmatrix} \Rightarrow \vec{M} = \begin{pmatrix} 1 \\ -9\frac{1}{2} \\ 7 \\ -12\frac{1}{2} \end{pmatrix}$$

4.2

Wendestellen

$$S(x)\Big|_{[x_{i-1},x_i]} = M_{i-1}\frac{(x_i - x)^3}{6h_i} + M_i\frac{(x - x_{i-1})^3}{6h_i} + C_i\left(x - \frac{x_{i-1} + x_i}{2}\right) + D_i \qquad \forall x \in [x_{i-1}, x_i]$$

$$\begin{aligned} S'(x)\Big|_{[x_{i-1},x_i]} &= M_{i-1}(-3)\frac{(x_i - x)^2}{6h_i} + M_i 3\frac{(x - x_{i-1})^2}{6h_i} + C_i \\ S''(x)\Big|_{[x_{i-1},x_i]} &= M_{i-1}6\frac{(x_i - x)}{6h_i} + M_i 6\frac{(x - x_{i-1})}{6h_i} \end{aligned}$$

$$\begin{aligned} \Rightarrow M_{i-1}x_i - M_{i-1}x + M_i x - M_i x_{i-1} &= 0 \\ x(M_i - M_{i-1}) &= M_i x_{i-1} - M_{i-1}x_i \\ x &= \frac{M_i x_{i-1} - M_{i-1}x_i}{M_i - M_{i-1}} \end{aligned}$$

$$\begin{aligned} i = 1 \Rightarrow x &= \frac{25}{27} \\ i = 2 \Rightarrow x &= \frac{15}{21} \\ i = 3 \Rightarrow x &= \\ i = 4 \Rightarrow x &= \end{aligned}$$

Spline

$$\begin{array}{ll} C_i = \frac{f[x_i]-f[x_{i-1}]}{h_i} - \frac{h_i}{6}(M_i - M_{i-1}) & D_i = \frac{f[x_i]-f[x_{i-1}]}{2} - \frac{h_i^2}{12}(M_i + M_{i-1}) \\ i = 1 \Rightarrow 0 = \frac{\frac{5}{2}-(-2)}{1} - \frac{1}{6}(7-(-20)) & \Rightarrow \frac{10}{3} \\ i = 2 \Rightarrow 6 & \Rightarrow \frac{7}{3} \\ i = 3 \Rightarrow -9 & \Rightarrow -\frac{7}{6} \\ i = 4 \Rightarrow 3 & \Rightarrow -\frac{2}{3} \end{array}$$

$$S(x)\Big|_{[0,1]} = -20\frac{(1-x)^3}{6} + 7\frac{(x-0)^3}{6} + 0(x-0) + \frac{10}{3}$$

5

5.1

5.2

6

6.1

6.2

6.3

6.4

Fehler in dieser Musterlösung bitte per e-mail an vordiplome@vmp.ethz.ch melden.

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