Problem sheet V

Fluid Dynamics 2005/2006

1. Shallow water waves and hydraulic theory

Let us consider the shallow water equations from Problem Sheet IV, i.e.

$$
\begin{aligned}
u_t + uu_x + vu_y - fv &= -g^* h_x \\
v_t + uv_x + vv_y + fu &= -g^* h_y \\
h_t + uh_x + vh_y + hu_x + hv_y &= 0
\end{aligned}
$$

(a) Taking the non-dimensionalized equations

$$
\begin{aligned}
R_0[\tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} + \tilde{v}\tilde{u}_{\tilde{y}}] - \tilde{v} &= (\exists R_0)\tilde{h}_{\tilde{x}} \\
R_0[\tilde{v}_{\tilde{t}} + \tilde{u}\tilde{v}_{\tilde{x}} + \tilde{v}\tilde{v}_{\tilde{y}}] + \tilde{u} &= (\exists R_0)\tilde{h}_{\tilde{y}} \\
\tilde{h}_{\tilde{t}} + \tilde{u}\tilde{h}_{\tilde{x}} + \tilde{v}\tilde{h}_{\tilde{y}} + \tilde{h}\tilde{u}_{\tilde{x}} + \tilde{h}\tilde{v}_{\tilde{y}} &= 0
\end{aligned}
$$

where $R_0 = \frac{U}{fL}$, $\exists = -g^*\left(\frac{H}{U^2}\right)$ and $g^* = \frac{\rho_1 - \rho_2}{\rho_1} g$. Show that the Coriolis force can be neglected for a system with: $U \sim 10$ m s^{-1}, $f \sim 10^{-4}$ s^{-1}, $L \sim 1$ m, $H \sim 10$ m and $\rho_1 >> \rho_2$ (e.g. water - air).

(b) Rewrite the dimensionalized equations without the Coriolis force and linearize with $u = \bar{u} + u'$, $h = \bar{h} + h'$ and $v = 0$ and show that h' is independent of y.

(c) Derive the wave equation for h' in the form:

$$h'_{tt} = \left(g^*\bar{h} - \bar{u}^2\right) h'_{xx} - 2\bar{u}h'_{tx}$$

(d) Find the dispersion relation for ω with $h' = \hat{h}' e^{i(kx - \omega t)}$, where $\hat{h}'$ is the amplitude of the wave.

(e) Show that the frequency ($\nu = \frac{\omega}{2\pi}$) of the waves propagating in the postive x-direction increases/decreases as $\bar{u}$ increases/decreases. This is known as Doppler Shift and can be observed for sound waves everytime a police car or ambulance passes you with the siren turned on.

(f) Show that for the waves obtained in (1d) $c_{ph} = c_g$ holds, where c_{ph} is the phase velocity and c_g is the group velocity. Note that for $\bar{u} = 0$ you will get the phase speed of linear shallow water gravity waves without a background flow ($c_{ph} = \sqrt{g^*\bar{h}}$).

(g) When orography has to be included in the shallow water equations the linearized equations from (1b) yield:

$$
\begin{aligned}
u'_t + \bar{u}u'_x &= -g^* h'_x && (1) \\
h'_t + \bar{u}\left(h'_x - h^b_x\right) + \bar{h}u'_x &= 0 && (2)
\end{aligned}
$$

where h^b is a function of x only and represents the height of orography.
Assuming steady-state derive an equation for u'_x in the form:

$$u'_x = \frac{\frac{g^*}{\bar{u}} Fr^2}{(1 - Fr^2)} h^b_x$$

where $Fr = \frac{\bar{u}}{\sqrt{g^*\bar{h}}} = \frac{\bar{u}}{c_{ph}}$ is the Froude number.

(... sheet continues on next page ...)

(h) Assuming a single hill in the flow domain draw a sketch for supercritical flow ($Fr > 0$) and subcritical flow ($Fr < 0$), respectively (nothing needs to be calculated here!). The sketch should include h as well as indications if u is greater or less than $\bar{u}$. (Hint: To be able to draw h you will need to utilize equation 1. Best if you define regions in your sketch: flow before reaching hill, flow above hill, flow after hill.)

(i) Draw a sketch also for the case where we observe a transition of flow regimes at the top of the hill from subcritical to supercritial flow. (Hint: Define regions in your sketch: flow before reaching hill, height of hill increasing, height of hill decreasing, flow after hill.) This flow setup is called 'shooting flow', due to the large u-velocities observed immediately behind the obstacle. Similar flows are observed during Föhn or Bora events or in shallow rivers with larger rocks underneath the water surface.

2. More linear shallow-water waves

Verify that the linear shallow-water-system

$$\begin{aligned} u'_t - fv' &= -g^* h'_x \\ v'_t + fu' &= -g^* h'_y \\ h'_t + \bar{h}(u'_x + v'_y) &= 0 \end{aligned}$$

has a solution (valid in the domain $y > 0$) of the form:

$$\begin{aligned} v' &= 0 \\ h' &= \hat{h'}(y) \sin(kx - \omega t) \\ u' &= \hat{u'}(y) \sin(kx - \omega t) \end{aligned}$$

where $\hat{h'}(y)$ and $\hat{u'}(y)$ are the amplitutes (only varying in the y-direction) of h' and u', respectively.

(a) Derive the wave-equation for h', determine the dispersion relation and evaluate the phase speed c_{ph}.

(b) Show that $\hat{u'}(y) = \frac{g^*}{c_{ph}} \hat{h'}(y)$.

(c) Deduce that $\hat{h'}(y) = Ae^{-\mu y}$ is one solution, where A is now the amplitude of $\hat{h'}(y)$ and determine the decay rate ($1/\mu$) as a function of $\bar{h}$, g^*, and f.

(d) Determine c_{ph} and $1/\mu$ for $\bar{h} \sim 10$ m, $f \sim 10^{-4}$ s^{-1} and $\rho_1 >> \rho_2$.

(e) What would be the maximum velocity perturbation u' with the assumed values of (2d) and $A = 1$ m (i.e. a wave with maximum Amplitude of 1 m).

(f) Taking the same values as in (2d), calculate the amplitude of a wave with perturbation velocity $u' = 10$ m s^{-1}? Does the linearization for our shallow water system still hold for this wave?

Due: Thursday, January 19, 2005, 3:00pm.
Either drop-off in office (P 15.1) or pigeon hole 'Spengler' in front of O 12.1.