UWIS, Fluid Dynamics, Problem sheet 4

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1 2D Flow field

$$\psi = A \sin(kx) \exp(-ly)$$

1.1 Sketch

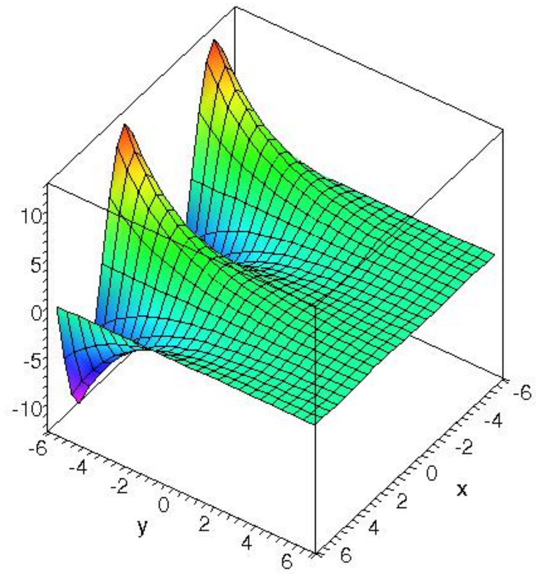
```
> restart;
> with(plots):
Warning, the name changecoords has been redefined
> psi := (x,y)->A * sin(k*x) * exp(-l*y);
```

$$(x, y) \mapsto \sin(x)\, e^{-0.4\, y} := (x, y) \mapsto \sin(x)\, e^{-0.4\, y}$$

```
> A := 1;
k := 1;
l := 0.4;
```

$$A := 1$$
$$k := 1$$
$$l := 0.4$$

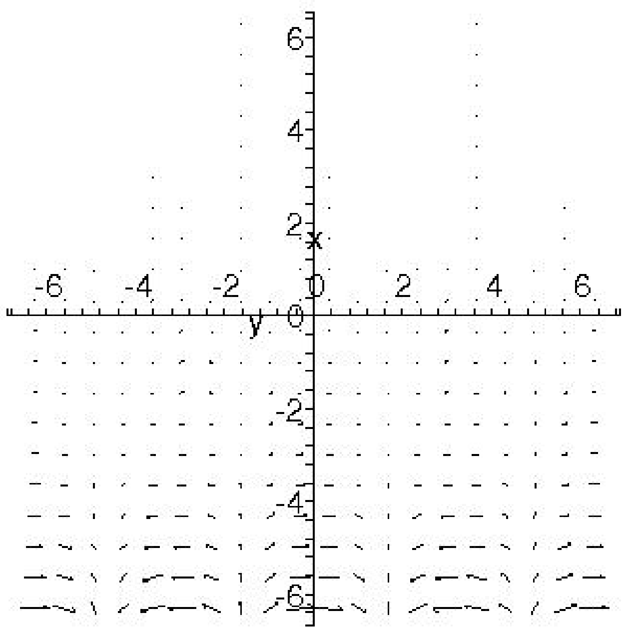
```
> plot3d(psi(x,y), x=-2*Pi..2*Pi, y=-2*Pi..2*Pi);
```



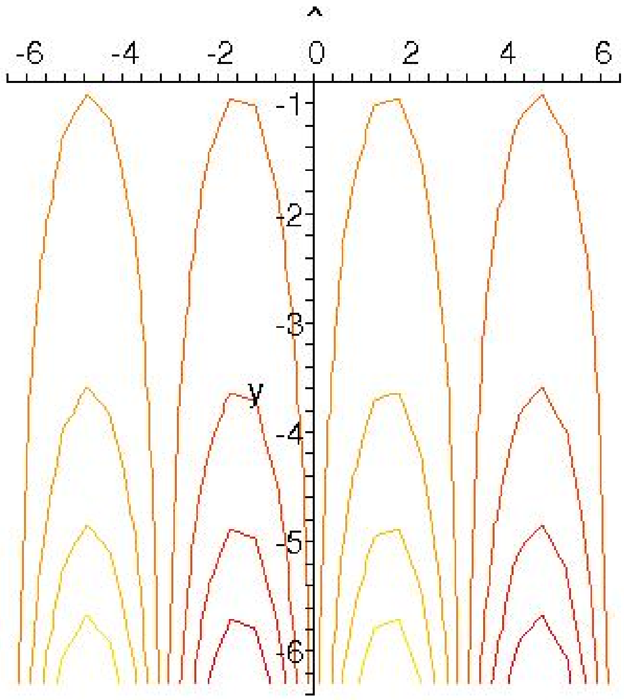
> with(plots):

Warning, the name changecoords has been redefined

> gradplot(psi(x,y), x=-2*Pi..2*Pi, y=-2*Pi..2*Pi);



> contourplot(psi(x,y), x=-2*Pi..2*Pi, y=-2*Pi..2*Pi);



1.2 Horizontal velocity

$$\frac{\partial}{\partial y}\psi = -Al\sin(kx)exp(-ly) = u$$
$$\frac{\partial}{\partial x}\psi = Ak\cos(kx)\exp(-ly) = v$$

1.3 Vorticity

$$\zeta = \frac{\partial}{\partial x}v - \frac{\partial}{\partial y}u \quad (1)$$

$$\frac{\partial}{\partial x}v = -Ak^2\sin(kx)\exp(-ly) \quad (2)$$
$$\frac{\partial}{\partial y}u = -Al^2\sin(kx)\exp(-ly) \quad (3)$$

$$(1) \overset{(2)\&(3)}{\Rightarrow} \zeta = (l^2-k^2)\underbrace{A\sin(kx)\exp(-ly)}_{\psi} \quad (4)$$

1.4 Streamfunction as a solution of a differential equation

$$\frac{D}{Dt}\zeta = \nu\Delta\zeta \tag{5}$$

$$\frac{D}{Dt}\zeta - \nu\Delta\zeta = 0 \quad \Big|\zeta \overset{(4)}{=} (l^2-k^2)\psi \tag{6}$$

$$\frac{D}{Dt}(l^2-k^2)\psi - \nu\Delta\left((l^2-k^2)\psi\right) = 0 \tag{7}$$

$$\underbrace{(l^2-k^2)}_{=0\Rightarrow l=\pm k}\left(\frac{D}{Dt}\psi - \nu\Delta\psi\right) = 0 \tag{8}$$

$$\frac{D}{Dt}\psi - \nu\Delta\psi = 0 \tag{9}$$

$$\vdots = \vdots \tag{10}$$

Now it is too late for good brain work, but here is the effort before Xmas. Use ψ as basic approach for ζ:

$$\begin{aligned}
\frac{\partial^2}{\partial x^2}u &= -A\cos(kx)k^3\exp(-ly)\\
\frac{\partial^2}{\partial y^2}u &= A\cos(kx)kl^2\exp(-ly)\\
\frac{\partial^2}{\partial x^2}v &= A\sin(kx)k^2 l\exp(-ly)\\
\frac{\partial^2}{\partial y^2}v &= -A\sin(kx)l^3\exp(-ly)
\end{aligned}$$

$$\Rightarrow \begin{pmatrix} \frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u \\ \frac{\partial^2}{\partial x^2}v + \frac{\partial^2}{\partial y^2}v \end{pmatrix} = \begin{pmatrix} -A\cos(kx)k^3\exp(-ly) + A\cos(kx)kl^2\exp(-ly) \\ A\sin(kx)k^2 l\exp(-ly) - A\sin(kx)l^3\exp(-ly) \end{pmatrix} =$$

$$\begin{pmatrix} A\cos(kx)\exp(-ly)k(-k^2+l^2) \\ A\sin(kx)\exp(-ly)l(k^2-l^2) \end{pmatrix}$$

$$\frac{D}{Dt} = 0 = \nu\begin{pmatrix} A\cos(kx)\exp(-ly)k(-k^2+l^2) \\ A\sin(kx)\exp(-ly)l(k^2-l^2) \end{pmatrix}$$

Tree possibilites

$$\nu\begin{pmatrix} -k^2+l^2 \\ k^2-l^2 \end{pmatrix} = 0 \Rightarrow k^2-l^2=0 \Rightarrow k^2=l^2 \Rightarrow k=\pm l$$

$$\begin{pmatrix} A\cos(kx)\exp(-ly)k(-k^2+l^2) \\ A\sin(kx)\exp(-ly)l(k^2-l^2) \end{pmatrix} = 0$$

$$\begin{pmatrix} k \\ l \end{pmatrix} = 0 \quad (trivial)$$

2 Flow through a tube: Law of Hagen Poiseuille

Navier-Stokes equation for the z-direction yields:

$$\frac{\partial}{\partial t}w + u\frac{\partial}{\partial r}w + \frac{v}{r}\frac{\partial}{\partial \varphi}w + w\frac{\partial}{\partial z}w = -\frac{1}{\varrho}\frac{\partial}{\partial z}p + \nu\Delta w$$

2.1 Rearrange

$$\underbrace{\frac{\partial}{\partial t}w}_{=0 \text{ (steady state)}} + \underbrace{u}_{=0}\frac{\partial}{\partial r}w + \underbrace{\frac{v}{r}}_{=0}\frac{\partial}{\partial \varphi}w + w\underbrace{\frac{\partial}{\partial z}w}_{=0} = -\frac{1}{\varrho}\frac{\partial}{\partial z}p + \nu\Delta w \tag{11}$$

$$0 = -\frac{1}{\varrho}\frac{\partial}{\partial z}p + \nu\Delta w \tag{12}$$

$$\frac{1}{\varrho}\frac{\partial}{\partial z}p = \nu\Delta w \tag{13}$$

$$\Delta w = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2} + \frac{\partial^2 w}{\partial z^2} \tag{14}$$

$$= \frac{1}{r}\left(\frac{\partial w}{\partial r} + r\frac{\partial^2 w}{\partial r^2}\right) + \underbrace{\frac{1}{r^2}\frac{\partial^2 w}{\partial \varphi^2}}_{=0 \text{ axial-symmetric}} + \underbrace{\frac{\partial^2 w}{\partial z^2}}_{=0 \text{ no source}} \tag{15}$$

$$(13) \overset{(15)}{\Rightarrow} \frac{1}{\varrho}\frac{\partial p}{\partial z} = \nu\frac{1}{r}\left(\frac{\partial w}{\partial r} + r\frac{\partial^2 w}{\partial r^2}\right) \tag{16}$$

2.2 Velocity profile

$$\frac{\partial p}{\partial z} = \text{const} = c \tag{17}$$

$$(16) \Rightarrow \frac{1}{\varrho}\underbrace{\frac{\partial p}{\partial z}}_{=c} = \nu\frac{1}{r}\frac{\partial w}{\partial r} + \nu\frac{\partial^2 w}{\partial r^2} \quad \Big| \cdot \frac{r}{\nu} \tag{18}$$

$$\frac{cr}{\varrho\nu} = \frac{\partial w}{\partial r} + r\frac{\partial^2 w}{\partial r^2} \quad \Big| \text{“from (15) to (14)”} \tag{19}$$

$$\frac{cr}{\varrho\nu} = \frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right) \quad \Big| \quad \underbrace{\int_0^r \mathrm{dr}}_{\text{integrate over r from 0 to r, that is unclean}} \tag{20}$$

$$\int_0^r \frac{cr}{\varrho\nu}\,\mathrm{dr} = \int_0^r \frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)\mathrm{dr} \tag{21}$$

$$\left.\frac{cr^2}{2\varrho\nu}\right|_0^r = \left. r\frac{\partial w}{\partial r}\right|_0^r \tag{22}$$

$$\frac{cr^2}{2\varrho\nu} = r\frac{\partial w}{\partial r} \quad \Big| \cdot \frac{\partial r}{r} \tag{23}$$

$$\frac{cr}{2\varrho\nu}\partial r = \partial w \quad \Big| \int \tag{24}$$

$$\int_0^r \frac{cr}{2\varrho\nu}\partial r = \int_{w(0)}^{w(r)} \partial w \tag{25}$$

$$\frac{cr^2}{4\varrho\nu} = w(r) - w(0) \tag{26}$$

$$\frac{cr^2}{4\varrho\nu} + w(0) = w(r) \quad \Big| \text{ boundary condition } w(R) = 0 \tag{27}$$

$$\frac{cR^2}{4\varrho\nu} + w(0) = 0 \tag{28}$$

$$w(0) = -\frac{cR^2}{4\varrho\nu} \tag{29}$$

$$(27) \overset{(29)}{\Rightarrow} w(r) = \frac{cr^2}{4\varrho\nu} - \frac{cR^2}{4\varrho\nu} \tag{30}$$

$$w(r) = \frac{c}{4\varrho\nu}(r^2 - R^2) \tag{31}$$

$$w(r) \overset{(17)}{=} \frac{1}{4\varrho\nu}\frac{\partial p}{\partial z}(r^2 - R^2) \tag{32}$$

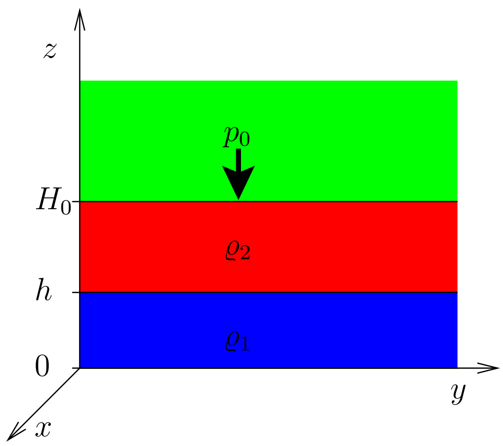
2.3 Mean flux

$$\int_0^{2\pi}\int_0^R w(r)\, r\mathrm{d}r\mathrm{d}\varphi =$$
$$\tfrac{1}{4\varrho\nu}\tfrac{\partial p}{\partial z}\int_0^{2\pi}\int_0^R (r^2-R^2)\, r\mathrm{d}r\mathrm{d}\varphi =$$
$$\tfrac{1}{4\varrho\nu}\tfrac{\partial p}{\partial z}\int_0^{2\pi}\int_0^R r^3 - R^2 r\,\mathrm{d}r\mathrm{d}\varphi =$$
$$\tfrac{1}{4\varrho\nu}\tfrac{\partial p}{\partial z}\int_0^{2\pi}\left(\tfrac{r^4}{4}-\tfrac{R^2r^2}{2}\right)\Big|_0^R \mathrm{d}\varphi =$$
$$\tfrac{1}{4\varrho\nu}\tfrac{\partial p}{\partial z}\int_0^{2\pi}\left(\tfrac{R^4}{4}-\tfrac{R^4}{2}\right)\mathrm{d}\varphi =$$
$$\tfrac{1}{4\varrho\nu}\tfrac{\partial p}{\partial z}\int_0^{2\pi} -\tfrac{R^4}{4}\,\mathrm{d}\varphi =$$
$$\tfrac{1}{4\varrho\nu}\tfrac{\partial p}{\partial z}\left(-\tfrac{R^4}{4}\varphi\right)\Big|_0^{2\pi} =$$
$$-\tfrac{1}{4\varrho\nu}\tfrac{\partial p}{\partial z}\tfrac{R^4}{4}2\pi =$$
$$-\tfrac{R^4\pi}{8\varrho\nu}\tfrac{\partial p}{\partial z}$$

2.4 Derive of the mean velocity

$$\begin{aligned}\bar{w} &= \frac{F}{A}\\ &= \frac{-\frac{R^4\pi}{8\varrho\nu}\frac{\partial p}{\partial z}}{R^2\pi}\\ &= -\frac{R^2}{8\varrho\nu}\frac{\partial p}{\partial z}\end{aligned}$$

3 Shallow water system



3.1 Pressure gradient force

$$
\begin{aligned}
\frac{\mathrm{dp}}{\mathrm{dz}} &= -\varrho g \\
\mathrm{dp} &= -\varrho g \mathrm{dz} \\
\int_{p(0)}^{p(h)} \mathrm{dp} &= \int_0^h -\varrho_1 g \,\mathrm{dz} \\
p(h) - p(0) &= -\varrho_1 g(h-0) \\
-p(0) &= -\varrho_1 gh - \underbrace{p(h)}_{(H_0-h)\varrho_2 g + p_0} \\
p(0) &= \varrho_1 gh + (H_0 - h)\varrho_2 g + p_0 \\
p(0) &= \varrho_1 gh - \varrho_2 gh + \varrho_2 g H_0 + p_0 \\
p(0) &= gh(\varrho_1 - \varrho_2) + \underbrace{\varrho_2 g H_0 + p_0}_{=0} \\
p(0) &= gh(\varrho_1 - \varrho_2) \\
\frac{p(0)}{\varrho_1} &= gh\frac{\varrho_1 - \varrho_2}{\varrho_1} \\
\frac{p(0)}{\varrho_1} &\overset{g^* = g\frac{\varrho_1 - \varrho_2}{\varrho_1}}{=} g^* h \\
\frac{1}{\varrho_1} p_x &= g^* h_x
\end{aligned}
$$

3.2 Non-dimensional

$$
\begin{aligned}
u_t + uu_x + vu_x - fv &= -g^* h_x & (33) \\
v_t + uv_x + vv_y + fu &= -g^* h_y & (34) \\
h_t + uh_x + vh_y + hu_x + hv_y &= 0 & (35)
\end{aligned}
$$

$$
\begin{aligned}
u &= \tilde{u}U \\
u_x &= \tilde{u}_{\tilde{x}}\frac{U}{L} \\
u_y &= \tilde{u}_{\tilde{y}}\frac{U}{L} \\
u_t &= \tilde{u}_{\tilde{t}}\frac{U^2}{L} \\
v &= \tilde{v}V = \tilde{v}U \\
v_x &= \tilde{v}_{\tilde{x}}\frac{V}{L} = \tilde{v}_{\tilde{x}}\frac{U}{L} \\
v_y &= \tilde{v}_{\tilde{y}}\frac{V}{L} = \tilde{v}_{\tilde{y}}\frac{U}{L} \\
h &= \tilde{h}H \\
h_x &= \tilde{h}_{\tilde{x}}\frac{H}{L} \\
h_y &= \tilde{h}_{\tilde{y}}\frac{H}{L} \\
h_t &= \tilde{h}_{\tilde{t}}\frac{HU}{L} \\
R_0 &= \frac{U}{fL} \Rightarrow R_0\frac{fL}{U} = 1
\end{aligned}
$$

$$
\begin{aligned}
(33) \Rightarrow u_t + uu_x + vu_y - fv &= -g^* h_x \\
\tilde{u}_{\tilde{t}}\frac{U^2}{L} + \tilde{u}U\tilde{u}_{\tilde{x}}\frac{U}{L} + \tilde{v}U\tilde{u}_{\tilde{y}}\frac{U}{L} - f\tilde{v}U &= -g^*\tilde{h}_{\tilde{x}}\frac{H}{L} \\
R_0\frac{fL}{U}\left(\tilde{u}_{\tilde{t}}\frac{U^2}{L} + \tilde{u}U\tilde{u}_{\tilde{x}}\frac{U}{L} + \tilde{v}U\tilde{u}_{\tilde{y}}\frac{U}{L}\right) - R_0\frac{fL}{U}f\tilde{v}U &= -g^*\tilde{h}_{\tilde{x}}\frac{H}{L}R_0\frac{fL}{U} \quad \Big| : f \\
R_0\left(\tilde{u}_{\tilde{t}}U + \tilde{u}U\tilde{u}_{\tilde{x}} + \tilde{v}U\tilde{u}_{\tilde{y}}\right) - R_0Lf\tilde{v} &= -g^*\tilde{h}_{\tilde{x}}R_0\frac{H}{U} \quad \Big| R_0 = \frac{U}{fL} \\
R_0U\left(\tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} + \tilde{v}\tilde{u}_{\tilde{y}}\right) - U\tilde{v} &= -g^*\tilde{h}_{\tilde{x}}R_0\frac{H}{U} \quad \Big| : U \\
R_0\left(\tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} + \tilde{v}\tilde{u}_{\tilde{y}}\right) - \tilde{v} &= \underbrace{-g^*\frac{H}{U^2}}_{=\exists}R_0\tilde{h}_{\tilde{x}}
\end{aligned}
$$

$$\begin{aligned}
(34) \Rightarrow v_t + uv_x + vv_y + fu &= -g^* h_y \\
R_0 \frac{fL}{U}\left(\tilde{v}_{\tilde{t}} \frac{U^2}{L} + \tilde{u}U\tilde{v}_{\tilde{x}}\frac{U}{L} + \tilde{v}U\tilde{v}_{\tilde{x}}\frac{U}{L}\right) + R_0\frac{fL}{U} f\tilde{u}U &= -g^*\tilde{h}_{\tilde{y}}\frac{H}{L}R_0\frac{fL}{U} \quad \Big| : f \\
R_0\left(\tilde{v}_{\tilde{t}}U + \tilde{u}U\tilde{v}_{\tilde{x}} + \tilde{v}U\tilde{v}_{\tilde{y}}\right) + R_0 Lf\tilde{u} &= -g^*\tilde{h}_{\tilde{y}}R_0\frac{H}{U} \quad \Big| R_0 = \frac{U}{fL} \\
R_0 U\left(\tilde{v}_{\tilde{t}} + \tilde{u}\tilde{v}_{\tilde{x}} + \tilde{v}\tilde{v}_{\tilde{y}}\right) + U\tilde{u} &= -g^*\tilde{h}_{\tilde{y}}R_0\frac{H}{U} \quad \Big| : U \\
R_0\left(\tilde{v}_{\tilde{t}} + \tilde{u}\tilde{v}_{\tilde{x}} + \tilde{v}\tilde{v}_{\tilde{y}}\right) + \tilde{u} &= \underbrace{-g^*\frac{H}{U^2}}_{=\exists} R_0\tilde{h}_{\tilde{y}}
\end{aligned}$$

$$\begin{aligned}
(35) \Rightarrow h_t + uh_x + vh_y + hu_x + hv_y &= 0 \\
\tilde{h}_{\tilde{t}}\frac{HU}{L} + \tilde{u}U\tilde{h}_{\tilde{x}}\frac{H}{L} + \tilde{v}U\tilde{h}_{\tilde{y}}\frac{H}{L} + \tilde{h}H\tilde{u}_{\tilde{x}}\frac{U}{L} + \tilde{h}H\tilde{v}_{\tilde{y}}\frac{U}{L} &= 0 \quad \Big| : \frac{HU}{L} \\
\tilde{h}_{\tilde{t}} + \tilde{u}\tilde{h}_{\tilde{x}} + \tilde{v}\tilde{h}_{\tilde{y}} + \tilde{h}\tilde{u}_{\tilde{x}} + \tilde{h}\tilde{v}_{\tilde{y}} &= 0
\end{aligned}$$

3.3 Dimensionless parameter

Why you use "∃" as a parameter? For me is "∃" reserved for "it exists".
From (3.2):

$$\exists = -g^*\frac{H}{U^2}$$

3.4 R_0 oft the observed motions

$$R_0 = \frac{U}{fL}$$

High and Lows	$R_0 = \frac{10}{10^{-4} \cdot 10^6}$ =	0.1	Coriolis effect plays the major rule
Hurricanes	$R_0 = \frac{10}{10^{-4} \cdot 10^5}$ =	1	$\frac{\mathrm{du}}{\mathrm{dt}}$ and coriolis effect are equated
Tornados	$R_0 = \frac{10^2}{10^{-4} \cdot 10^3}$ =	1000	$\frac{\mathrm{du}}{\mathrm{dt}}$ plays the major rule

3.5 ∃ for high and lows

$$\exists = -g^*\frac{H}{U^2} = -g^*\frac{H}{U^2} = -10\,\frac{\mathrm{m}}{\mathrm{s}^2}\frac{10^2\,\mathrm{m}}{10^2\,\frac{\mathrm{m}^2}{\mathrm{s}^2}} = -10$$