1. Atmospheric eddy

(a) Radial velocity v_c= $10\,\mathrm{ms}^{-1}$ at R, radius $R = 500\,\mathrm{km}$

- Circulation around a circle:

$$\begin{aligned} C = \int_0^{2\pi} v_c d_c &= 2\pi \cdot R \cdot v_c \\ &= 2\pi \cdot 5 \cdot 10^5 \mathrm{m} \cdot 10\,\mathrm{ms}^{-1} \\ &= \pi \cdot \mathbf{10^7} \mathrm{m}^{\mathbf{2}} \mathrm{s}^{\mathbf{-1}} \end{aligned}$$

- Mean vorticity:

$$\begin{aligned} \zeta &= \frac{C}{\pi R^2} = \frac{2\pi R v_c}{\pi R^2} = \frac{2v_c}{R} \\ &= \mathbf{4 \cdot 10^{-5}} \mathrm{s}^{\mathbf{-1}} \end{aligned}$$

(b) Vorticity in polar coordinates (see lecture notes page ??):

$$\zeta = \frac{1}{r}\frac{\partial}{\partial r}(rv) = \frac{v}{r} + \frac{\partial v}{\partial r}$$

Assuming that there is no vorticity in the region outside R, leads to:

$$\zeta = \begin{cases} \frac{2\,v_c}{R} & r = R \\ 0 & r > R \end{cases}$$

Derive the expression for $v(r)$ if $r > R$ (ξ must be 0):

$$\begin{aligned} \frac{v}{r} + \frac{\partial v}{\partial r} &= 0 \\ \frac{\partial v}{v} &= -\frac{\partial r}{r} \\ \ln v &= -\ln r + K \\ v(r) &= K \cdot e^{-\ln r} \\ v(r) &= \frac{K}{r} \end{aligned}$$

Here K is a constant. That means that the velocity decreases outside R with K/r! Derive the value of K with the boundary condition $v = v_c = 10\,\mathrm{ms}^{-1}$ if $r \to R$.

$$\begin{aligned} K &= v_c \cdot R \\ &= 5 \cdot 10^6\,\mathrm{m}^2\mathrm{s}^{-1} \end{aligned}$$

Now we can calculate the the velocity at radius 1000 km entierly induced by the vorticity within 500 km:

$$\begin{aligned} v(r) &= \frac{K}{r} = \frac{v_c \cdot R}{r} \\ v(1000\,\mathrm{km}) &= \frac{10\,\mathrm{ms}^{-1} \cdot 5 \cdot 15^5\,\mathrm{m}}{10^6\,\mathrm{m}} \\ &= \mathbf{5}\,\mathrm{ms}^{\mathbf{-1}} \end{aligned}$$

2. Two-dimensional vortex

Two velocity fields, one from the uniform flow v_1 and one from the two-dimensional vortex v_2.

- v_1: The streamfunction ψ in polar coordinates is given by $\psi = -U\,r\,\sin\Theta$ (in Cartesian coordinates $\psi = -U \cdot y$), therefore:

$$\begin{aligned} u_1 &= -\frac{1}{r}\frac{\partial\psi}{\partial\Theta} = U\,\cos\Theta \\ v_1 &= \frac{\partial\psi}{\partial r} = -U\,\sin\Theta \end{aligned}$$

- v_2:

$$v_2 = \frac{C}{2\,\pi\,a} = \frac{\Lambda\,\pi\,a^2}{2\,\pi\,a} = \frac{\Lambda\,a}{2}$$

Vortex in stationary flow:

$$v = v_1 + v_2 = \begin{pmatrix} U\cos\Theta \\ -U\sin\Theta + \frac{\Lambda\,a}{2} \end{pmatrix}$$

The stagnation point is located where $v = 0$ ($U = \frac{\Lambda\,a}{2}$):

$$v = \frac{\Lambda\,a}{2}\begin{pmatrix} \cos\Theta \\ \sin\Theta + 1 \end{pmatrix} = 0$$

Therefore:

$$\cos\Theta = 0$$
$$\sin\Theta = 1$$

That means that the stagnation point is located at the edge of the vortex ($r = a$) at $\Theta = \pi/2$.

3. Rossby and Ekman numbers

Rossby number: $R_0 = \frac{U}{fL}$ $\quad\left(\frac{\text{velocity}}{\text{coriolis}}\right)$

Ekman number: $E = \frac{\nu}{fL^2}$ $\quad\left(\frac{\text{friction}}{\text{coriolis}}\right)$

	L [m]	U [ms^{-1}]	ν [m^{2}s^{-1}]	R_0	E
air	10^6	10^1	$1.5 \cdot 10^{-5}$	10^{-1}	$1.5 \cdot 10^{-13}$
water	10^5	10^{-2}	$1.0 \cdot 10^{-6}$	10^{-3}	$1.0 \cdot 10^{-12}$

R_0 and $E << 1$ for atmospheric and oceanic flows. That means the inertial (U) and friction (ν) effects are small compared with the Coriolis force.

Remark: $R_{0_{\text{air}}}$ is larger than $R_{0_{\text{water}}}$. The inertial effects are more important in the atmosphere than int the ocean.

Navier-Stokes equation (see lecture notes page 101):

$$R_0 \frac{Dv}{Dt} - \left(\vec{k} \times \vec{v}\right) f = -\frac{1}{\rho} \vec{\nabla} p + E \nabla^2 \nu$$

With R_0 = Rossby number and E = Ekman number.
For large scale flows (synoptic scale) the Rossby- and Ekman numbers are << 1 as seen in the solution above. Therefore term (1) and (4) can be neclected. A geostrophic balance exists between the Coriolis- and the pressure gradient force.

$$\begin{aligned} fv &= \frac{1}{\rho} \frac{\partial p}{\partial x} \\ fu &= -\frac{1}{\rho} \frac{\partial p}{\partial y} \end{aligned}$$

If we don't split the equation in a meridional and zonal component we get:

$$f\vec{v} = \frac{1}{\rho} \frac{\partial p}{\partial \tilde{x}}$$

Geostrophic velocity:

$$\begin{aligned} \vec{v} &= \frac{1}{\rho f} \frac{\partial p}{\partial \tilde{x}} \\ &= \frac{1}{1\,\mathrm{kg\,m^{-3}} \cdot 10^{-4}} \frac{10^3\,\mathrm{Pa}}{10^6\,\mathrm{m}} \\ &\sim \mathbf{10\,ms^{-1}} \end{aligned}$$