Solutions to problem sheet V

1. Tree

(a) Linear shallow-water wave theory:

Dispersion relationship:

$$\omega\left(-\omega^2 + f^2 + l^2\left(gH\right)\right) = 0$$

The phase velocity is dependent upon the mean depth H:

$$\begin{aligned}
V_p &= \frac{\omega}{l} = \frac{\sqrt{f^2 + l^2\left(gH\right)}}{l} \\
V_p &= \sqrt{g \cdot H} \\
V_p &= V_g = \sqrt{10\,\mathrm{ms}^{-2} \cdot 3.6\,\mathrm{m}} = 6\,\mathrm{ms}^{-1} \\
\Rightarrow t &= \frac{d}{v} = \frac{100\,\mathrm{m}}{6\,\mathrm{ms}^{-1}} \sim 16\,\mathrm{s}
\end{aligned}$$

The disturbance will be felt 100 m away after 16 s.

(b)

$$v' = \bar{u} \pm v \qquad \bar{u} : \text{mean flow velocity}$$

downstream:

$$\text{(i)} \qquad \bar{u} = 3\,\mathrm{ms}^{-1} \Rightarrow \begin{cases} v_1' = 9\,\mathrm{ms}^{-1} \\ t_1' = 11\,\mathrm{s} \end{cases}$$

$$\text{(ii)} \qquad \bar{u} = 6\,\mathrm{ms}^{-1} \Rightarrow \begin{cases} v_1' = 12\,\mathrm{ms}^{-1} \\ t_1' \approx 8\,\mathrm{s} \end{cases}$$

upstream:

$$\text{(i)} \qquad \bar{u} = 3\,\mathrm{ms}^{-1} \Rightarrow \begin{cases} v_2' = 3\,\mathrm{ms}^{-1} \\ t_2' = 33\,\mathrm{s} \end{cases}$$

$$\text{(ii)} \qquad \bar{u} = 6\,\mathrm{ms}^{-1} \Rightarrow \begin{cases} v_2' = 0\,\mathrm{ms}^{-1} \\ t_2' = \text{never} \end{cases}$$

where v_1' propagation velocity downstream and v_2' propagation velocity upstream

2. Lake

Shallowness: horizontal and vertical characteristic length scales such that $\delta = \frac{H}{L} \ll 1$

(a) Here H=10m, L=10'000m
$\Rightarrow \delta = 10^{-2} \ll 1$
linear shallow-water theory: f is zero, rotation effect neglected, reflection case (b) (script p. 70)
Lake configuration, flow confined at $(-\mathrm{L}, 0)$, such that boundary conditions:
$\sin(lL) \stackrel{!}{=} 0 \quad \mu \in ??$
$\Rightarrow l_\mu = \frac{\mu\pi}{L}$, l only takes on DISCRETE values

WANTED: periods of the natural oscillation

$$
\begin{aligned}
T_\mu &= \frac{2\pi}{\omega_\mu} \text{ where } \omega_\mu = c l_\mu \qquad \left(\text{since } \ c = V_p = \frac{\omega}{l}\right) \\
c &= \sqrt{gH} = 10\mathrm{ms}^{-1}
\end{aligned}
$$

$$
l_1 = \frac{\pi}{L}, \ l_2 = \frac{2\pi}{L} \ \ldots
$$

$$
\begin{aligned}
\Rightarrow T_1 &= \frac{2\pi}{cl_1} = \frac{2L}{c} = \frac{2 * 10'000}{10} = 2 * 10^3\mathrm{s} \sim 33\,\mathrm{min} \\
\Rightarrow T_2 &= \frac{T_1}{T_2} = 1000\mathrm{s} \sim 16\mathrm{min} \\
\Rightarrow T_3 &= \frac{T_1}{3} \sim 11\,\mathrm{min}
\end{aligned}
$$

(b) longest natural oscillation $\approx$ main tide period

$$
\begin{aligned}
T_{tide} &= 12\mathrm{h} \stackrel{!}{=} T_1 \\
&\Rightarrow \ T_1 = \frac{2\pi}{c}\frac{L_{lagoon}}{\pi} \Rightarrow L_{lagoon} = \frac{T_1 c}{2} \\
&\Rightarrow \ L_{lagoon} = \frac{12 * 60^2 * 10}{2} \approx 216\,\mathrm{km}
\end{aligned}
$$

longest natural oscillation and main tide have approximately the same period
$\Rightarrow$ RESONANCE flow response.

3. Linear shallow water system

$$
v' = 0 \Rightarrow \begin{cases} u'_t = -gh'_x & \text{(1) height variation} \rightarrow \text{flow acceleration} \\ fu' = -gh'_y & \text{(2) coriolis repartition} \rightarrow \text{height variation} \\ h'_t + Hu'_x = 0 & \text{(3) flow divergence} \rightarrow \text{height change} \end{cases}
$$

$(3)_t$

$$
\frac{\partial^2 h'}{\partial t^2} + \frac{\partial^2 u'}{\partial x \partial t}
$$

$(1)_x$

$$
\frac{\partial^2 u'}{\partial t \partial x} = -g\frac{\partial^2 h'}{\partial x^2}
$$

$(3)_t + (1)_x$

$$\Rightarrow \frac{\partial^2 h'}{\partial t^2} - gH\frac{\partial^2 h'}{\partial x^2} = 0 \quad \text{(Wellengleichung)}$$

For $h' = A\sin\left(k\left(x-ct\right)\right)e^{-\mu y}$ and $u' = \left(\frac{c}{H}\right)h'$

$$\begin{aligned}
\frac{\partial h'}{\partial t} &= -kcA\cos\left(k\left(x-ct\right)\right)e^{-\mu y} \\
\frac{\partial^2 h'}{\partial t^2} &= -\left(kc\right)^2 h' \\
\frac{\partial h'}{\partial x} &= kA\cos\left(k\left(x-ct\right)\right)e^{-\mu y} \\
\Rightarrow \frac{\partial^2 h'}{\partial x^2} &= -\left(k\right)^2 h' \\
\Rightarrow -\left(kc\right)^2 h' &= gH\left(-k^2\right)h' \\
c^2 &= gH \\
c &= \sqrt{gH}
\end{aligned}$$

Back to equation (2):

$$\begin{aligned}
\frac{\partial h'}{\partial y} &= -\mu h' \\
\Rightarrow fu' &= \mu g h'
\end{aligned}$$

since

$$\begin{aligned}
u' &= \left(\frac{c}{H}\right) \\
f\left(\frac{c}{H}\right) &= \mu g \\
\mu &= \frac{fc}{gH} \\
\mu &= \frac{f}{c} \\
\frac{1}{\mu} &= \frac{\sqrt{gH}}{f}
\end{aligned}$$

$$\begin{aligned}
u'_{max} &= \frac{1}{f}g\mu A\sin(k(x-ct))e^{\mu y} \\
&= \frac{g\mu}{f}A \\
&= \frac{10\cdot 10^{-5}}{10^{-4}} = A
\end{aligned}$$