1. Laminar flow in a water channel

Balance between gravitational force and vicous stress:

$$u\left(z\right) = \Gamma H z - \frac{1}{2}\Gamma z^2 \qquad \text{where } \Gamma = g \cdot \frac{\sin\alpha}{\nu}$$

Discharge of water:

$$Q = \int_0^H \rho \cdot u\left(z\right) dz$$

(a) What is the percentage change in depth, if the discharge is increased by 20%:
First derive the discharge:

$$\begin{aligned} Q &= \int_0^H \rho \left[\Gamma H z - \frac{1}{2}\Gamma z^2\right] dz \\ &= \rho \left[\frac{1}{2}\Gamma H^3 - \frac{1}{6}\Gamma H^3\right] \\ &= \frac{1}{3}\rho\Gamma H^3 \end{aligned}$$

Get now the change in discharche by 20%:

$$\begin{aligned} Q' &= 1.2 \cdot Q \\ \frac{1}{3}\rho\Gamma H'^3 &= 1.2 \cdot \frac{1}{3}\rho\Gamma H^3 \\ H'^3 &= 1.2 \cdot H^3 \\ H' &= \sqrt[3]{1.2} \\ H' &\sim 1.063\, H \end{aligned}$$

The height of the water channel increases by 6.3%.

(b) The change of depth is neither dependent upon the viscosity nor the temperature. Therefore the 6.3% change is valid whatever the fluid properties are.

2. Water flow

One solution of the governing equations is:

$$u\left(z\right) = U\left(1 - e^{-\frac{z}{\varepsilon}}\right)$$

Boundary layer: Layer over which the effects of the surface (e.g. frictional effects) can be felt.

$$\epsilon = \frac{\nu}{W} \qquad \text{where} \begin{cases} \nu \text{ dynamical viscosity} \\ W = 10^{-2} \text{ms}^{-1} \end{cases}$$

	ν [m^2s^{-1}]	ε [m]
Water	$1.1 \cdot 10^{-6}$	$1.1 \cdot 10^{-4} = 0.1$ mm
Air	$1.5 \cdot 10^{-5}$	$1.5 \cdot 10^{-3} = 1.5$ mm

So the boundary layer of the atmosphere is generally speaking much deeper than that of the ocean.

3. 2D flow field

Streamfunction:

$$\psi = A\sin\left(kx\right)e^{-ly}$$

(a) Streamfunction and streamlines

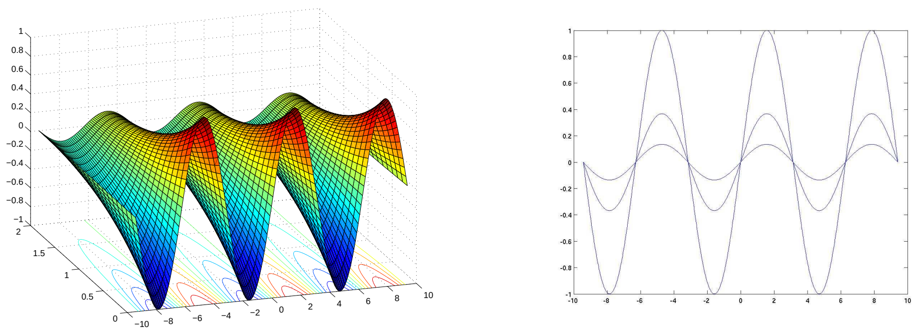
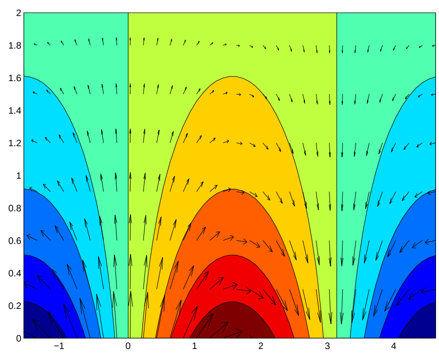


Figure 1: Streamfunction (left) and the some corresponding streamlines (right).

(b) Velocity components u and v:

$$u = -\frac{\partial\psi}{\partial y} = Al\sin\left(kx\right)e^{-ly}$$
$$v = -\frac{\partial\psi}{\partial x} = Ak\cos\left(kx\right)e^{-ly}$$



(c) Vertical vorticity:

$$\begin{aligned}
\zeta &= \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \\
\frac{\partial v}{\partial x} &= -Ak^2 \sin(kx)\, e^{-ly} \\
\frac{\partial u}{\partial y} &= -Al^2 \sin(kx)\, e^{-ly} \\
\zeta &= \left(l^2 - k^2\right) A \sin(kx)\, e^{-ly} = \left(l^2 - k^2\right) \psi
\end{aligned}$$

(d) Values of k and l to be a solution of:

$$\frac{D\zeta}{Dt} = \nu \cdot \nabla^2 \zeta$$

$$\begin{aligned}
\frac{D}{Dt}(l^2 - k^2)\psi &= \nu \nabla^2 (l^2 - k^2)\psi \\
(l^2 - k^2)\left[\frac{D\psi}{Dt} - \nu \nabla^2 \psi\right] &= 0
\end{aligned}$$

Two possibilities:

$$\begin{aligned}
(l^2 - k^2) &= 0 \\
l &= \pm k \\
&\text{or} \\
\frac{D\psi}{Dt} - \nu \nabla^2 \psi &= 0 \\
k &= 0 \text{ (trivial solution)}
\end{aligned}$$

4. Shallow water system

Set of equations (6.16) in lecture notes on page 65. 1-dimensional variation in y, z direction $\frac{\partial}{\partial y}, \frac{\partial}{\partial z} = 0$.

(a) Normalisation and parameter ∃

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - fv = -g\frac{\partial h}{\partial x} \quad (1)$$

$$\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} - fu = 0 \quad (2)$$

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + h\frac{\partial u}{\partial x} = 0 \quad (3)$$

Normalize with: $\tilde{u} = \frac{u}{U}$, $\tilde{h} = \frac{h}{H}$, $\tilde{x} = \frac{x}{L}$, $\tilde{t} = \frac{U}{L}t$

- (1)

$$\begin{aligned}\frac{\partial u}{\partial t} &= \frac{U^2}{L}\frac{\partial \tilde{u}}{\partial \tilde{t}}, \qquad u_t = \frac{U^2}{L}\tilde{u}_{\tilde{t}} \\ \frac{\partial u}{\partial x} &= \frac{U}{L}\frac{\partial \tilde{u}}{\partial \tilde{x}}, \qquad u_x = \frac{U}{L}\tilde{u}_{\tilde{x}} \\ \frac{\partial h}{\partial x} &= \frac{H}{L}\frac{\partial \tilde{h}}{\partial \tilde{x}}, \qquad h_x = \frac{H}{L}\tilde{h}_{\tilde{x}}\end{aligned}$$

$$\frac{U^2}{L}\left(\tilde{u}_{\tilde{t}} + \tilde{u}_{\tilde{x}}\right) - fU\tilde{v} = -g\frac{H}{L}\tilde{h}_{\tilde{x}}$$

With $R_0 = \frac{U}{fL}$ follows:

$$\begin{aligned}R_0\left(Uf\left(\tilde{u}_{\tilde{t}} + \tilde{u}_{\tilde{x}}\right)\right) - fU\tilde{v} &= R_0\left(-gf\frac{H}{U}\right)\tilde{h}_{\tilde{x}} \\ R_0\left(\tilde{u}_{\tilde{t}} + \tilde{u}_{\tilde{x}}\right) - \tilde{v} &= R_0\left(-g\frac{H}{U^2}\right)\tilde{h}_{\tilde{x}}\end{aligned}$$

Therefore: $\exists = -g\left(\frac{H}{U^2}\right)$

- (2)

$$\begin{aligned}\frac{U^2}{L}\tilde{v}_{\tilde{t}} + \frac{U^2}{L}\tilde{u}\tilde{v}_{\tilde{x}} + fU\tilde{u} &= 0 \\ R_0\left(fU\tilde{v}_{\tilde{t}} + fU\tilde{u}\tilde{v}_{\tilde{x}}\right) + fU\tilde{u} &= 0 \\ R_0\left(\tilde{v}_{\tilde{t}} + \tilde{u}\tilde{v}_{\tilde{x}}\right) + \tilde{u} &= 0\end{aligned}$$

- (3)

$$\begin{aligned}\frac{UH}{L}\tilde{h}_{\tilde{t}} + \frac{UH}{L}\tilde{h}_{\tilde{x}}\tilde{u} + \frac{UH}{L}\tilde{u}_{\tilde{x}}\tilde{h}u &= 0 \\ \tilde{h}_{\tilde{t}} + \tilde{h}_{\tilde{x}}\tilde{u} + \tilde{u}_{\tilde{x}}\tilde{h}u &= 0\end{aligned}$$

(b) Characteristic values

$$\begin{aligned}R_0 &= \frac{U}{fL} = \frac{10^{-2}\,\mathrm{ms}^{-1}}{10^{-4}\,\mathrm{s}^{-1}10^{4}\,\mathrm{m}} = 10^{-2} \\ &\rightarrow \text{Corioliseffect is important} \\ \exists &= -\frac{gH}{U^2} = \frac{10\,\mathrm{ms}^{-2}\cdot 10\,\mathrm{m}}{\left(10^{-2}\,\mathrm{ms}^{-1}\right)^2} = 10^{6}\end{aligned}$$

(c) Dynamical similarity

The combination of some key elements is more important than elements themselves separately. Different configuration but same combination $\Rightarrow$ same flow response similar for same $\exists$, R_0 as (b).

Assumptions: $R_0 = 10^{-2}$, $\exists = 10^6$
Lake: $U_L = 10^{-2}$ ms^{-1}, $L_L = 10^4$ m
Model: $U_M = 10^{-3}$ ms^{-1}, $L_M = 1$ m

Depth H_{Model},f_{Model} =?

$$f_{\text{Model}} = \frac{U_M}{R_0 L_M} = \frac{10^{-3}\,\text{ms}^{-1}}{10^{-2}\cdot 1\,\text{m}} = 10^{-1}\,\text{s}^{-1}$$

Since $f = 2\omega \Rightarrow \omega = 5\cdot 10^{-2}\text{s}^{-1}$
$\Rightarrow T = \frac{2\pi}{\omega} \approx 126\text{s} \sim 2\text{min}$

$$H_{\text{Model}} = \frac{\exists U_M^2}{g} = \frac{10^6\cdot\left(10^{-3}\text{ms}^{-1}\right)^2}{10\text{ms}^{-2}} = 10^{-1}\,\text{m}$$