Solutions to problem sheet III

1. Pressure

Pressure gradient force:

$$F_G = -\frac{1}{\rho}\vec{\nabla}p$$

Due to:

$$\begin{aligned} F &= m \cdot a \\ F &= dp \cdot A \\ m &= A \cdot dz \cdot \rho \end{aligned}$$

With $A =$ area and $\rho =$ density.
Acceleration:

$$\frac{Dv}{Dt} = -\frac{1}{\rho}\vec{\nabla}p = -\frac{1}{1\text{kgm}^{-3}}\frac{10^3\text{Pa}}{2 \cdot 10^6\text{m}} \cong 5 \cdot 10^{-4}\text{ms}^{-2}$$

Velocity after 1 hour:

$$v = a \cdot t = 5 \cdot 10^{-4}\text{ms}^{-1} \cdot 3600\text{s} = 1.8\text{ms}^{-1} \cong 6.5\text{kmh}^{-1}$$

Although pressure gradient not that strong (10Pa/2000km), acceleration taking place still significantly.

2. Snooker

Explanation coriolis force:

$$\vec{F_c} = -2\left(\vec{\Omega} \wedge \vec{v}\right) \text{ with } \vec{\Omega} = \begin{pmatrix} 0 \\ \Omega\cos\varphi \\ \Omega\sin\varphi \end{pmatrix} \text{ and } \vec{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

$$\vec{\Omega} \wedge \vec{v} = \begin{pmatrix} -\Omega\sin\varphi v + \Omega\cos\varphi w \\ u\Omega\sin\varphi \\ -u\Omega\cos\varphi \end{pmatrix}$$

2-dimensional acceleration $\Rightarrow w = 0$:

$$\vec{F_c} = -2\Omega \begin{pmatrix} -v\sin\varphi \\ u\sin\varphi \\ -u\cos\varphi \end{pmatrix}$$

Coriolis parameter: $f = 2\Omega\sin\varphi \cong 10^{-4}$ at midlatitudes.

(a) $t = 3\text{s}$, $v = 1\text{ms}^{-1}$

Coriolis acceleration: $\frac{Du}{Dt} = fv$

Displacement: $x = \frac{1}{2} fvt^2 \cong 5 \cdot 10^{-4}\text{m} \cong 0.5\text{mm}$

or

Coriolis acceleration: $\frac{Du}{Dt} = 2\,(\Omega \vec{v})$

in horizontal plane: $\frac{Du}{Dt} = 2 \cdot \Omega \vec{v} \cdot \sin\alpha$

Displacement: $x = \frac{1}{2} \frac{Du}{Dt} t^2 \cong 5 \cdot 10^{-4}\text{m} \cong 0.5\text{mm}$

(b) Northern hemisphere: $\vec{F}_c \perp \vec{v}$
Facing to the velocity vector, the coriolis force is aimed to the right.

Southern hemisphere: $\vec{F}_c \perp \vec{v}$ but: $\vec{\Omega} \mapsto -\vec{\Omega}$
Facing to the velocity vector, the coriolis force is aimed to the left.

Athletic track races are always anticlockwise (left turn). Therefore the race in Melbourne (southern hemisphere) is backed by $\vec{F}_c$, in Zurich it is the opposite, respectively.

3. "Fun-fair" ride

The centrifugal force must be stronger than the gravitational force:

$$
\begin{aligned}
F_{cent} > g \quad &\Rightarrow \quad r\omega^2 > g \\
\omega \quad &> \quad \sqrt{\frac{g}{r}} \\
&> \quad \sqrt{\frac{9.81\,\text{ms}^{-2}}{6\,\text{m}}} \\
&> \quad 1.28\,\text{rad s}^{-1}
\end{aligned}
$$

or

$$
\begin{aligned}
r\omega^2 &= \frac{v^2}{r} > g \\
v &= \sqrt{g \cdot r} = 7.7\text{ms}^{-1}
\end{aligned}
$$

The time of circulation is: $\tau = \frac{2\pi}{\omega}$

$$
\tau < \frac{2\pi}{1.28\,\text{rad s}^{-1}} < 4.9\,\text{s}
$$

4. Adiabatic temperature change

Typical example of fall wind (e.g."Fohn$''$, Chinook) due to adiabatic compression. Compare script p. 33.

$$\frac{DH}{Dt} = c_p dT - \alpha dp = 0 \qquad \alpha = \frac{1}{\rho} = \frac{R^*T}{p}$$

$$c_p \frac{dT}{T} = R^* \frac{dp}{p}$$

$$c_p \ln T|_{T_1}^{T_2} = R^* \ln p|_{p_1}^{p_2}$$

$$c_p (\ln T_1 - \ln T_2) = R^* (\ln p_1 - \ln p_2)$$

$$c_p \left(\ln \frac{T_1}{T_2}\right) = R^* \left(\ln \frac{p_1}{p_2}\right)$$

$$\ln \frac{T_1}{T_2} = \frac{R^*}{c_p} \cdot \ln \frac{p_1}{p_2}$$

$$\left(\frac{T_1}{T_2}\right) = \left(\frac{p_1}{p_2}\right)^{\frac{R^*}{c_p}}$$

If the flow is considered as adiabatic: $\frac{D\Theta}{Dt} = 0$. Therefore $\Theta_{Altdorf} = \Theta_{Gotthard}$.

$$T_G \left(\frac{p_0}{p_G}\right)^{\kappa} = T_A \left(\frac{p_0}{p_A}\right)^{\kappa}$$

$$\Rightarrow \frac{T_A}{T_G} = \left(\frac{p_A}{p_G}\right)^{\kappa}$$

Assumptions: $p_0 = 1000$ hPA, $p_A = 960$ hPa, $p_G = 780$ hPa, $T_A = 273$ K, $\kappa = \frac{R}{c_p} = 0.286$???

$$\frac{T_A}{T_G} = 1.06 \qquad \Rightarrow \qquad T_A = 289\,\mathrm{K}$$

$$\Delta T \cong 16\,\mathrm{K} \qquad \text{(only due to compression)}$$

5. Laplace equation

$$\nabla^2 \Phi = \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\partial^2 \Phi}{\partial z^2}$$

6. Cartesian coordinates:
Solve $\nabla^2 \Phi$ with:

$$\Phi = (x^2 + y^2 + z^2)^{-1/2}$$

$$\frac{\partial \Phi}{\partial x} = -\frac{x}{(x^2 + y^2 + z^2)^{3/2}}$$

$$\frac{\partial^2 \Phi}{\partial x^2} = -\frac{2x^2 - y^2 - z^2}{(x^2 + y^2 + z^2)^{5/2}}$$

$$
\begin{aligned}
\frac{\partial \Phi}{\partial x} &= -\frac{x}{(x^2+y^2+z^2)^{3/2}} \\
\frac{\partial^2 \Phi}{\partial x^2} &= -\frac{2x^2-y^2-z^2}{(x^2+y^2+z^2)^{5/2}}
\end{aligned}
$$

(a) Cylindrical polar coordinates:

$$
\begin{aligned}
\nabla^2 \Phi &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{\partial}{\partial \varphi}\left(\frac{1}{r}\frac{\partial \Phi}{\partial \varphi}\right) + \frac{\partial}{\partial z}\left(r\frac{\partial \Phi}{\partial z}\right) \\
&= \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \varphi^2} + \frac{\partial^2 \Phi}{\partial z^2}
\end{aligned}
$$

For $\Phi = \ln r$, only dependent on r:

$$
\begin{gathered}
\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} \\
-\frac{1}{r^2} + \frac{1}{r^2} = 0
\end{gathered}
$$