1. Streamlines

(a) Cartesian coordinates:

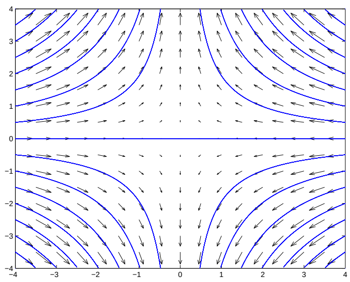
$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = a \begin{pmatrix} -x \\ y \\ 0 \end{pmatrix}$$

Streamlines:

$$\frac{dx}{u} = \frac{dy}{v}$$

Therefore:

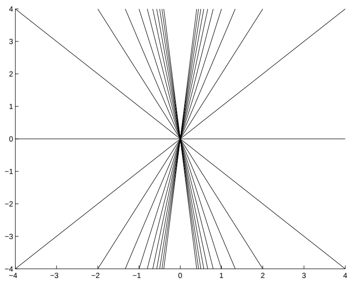
$$\begin{aligned} -\frac{dx}{x} = \frac{dy}{y} \Rightarrow \ln y &= \ln\frac{1}{x} + C \\ y &= \frac{C}{x} \qquad \text{(One streamline for any C)} \end{aligned}$$



(b) Cylindrical polar coordinates:

$$\begin{pmatrix} u_r \\ u_\varphi \\ u_z \end{pmatrix} = \begin{pmatrix} m/r \\ 0 \\ 0 \end{pmatrix}$$

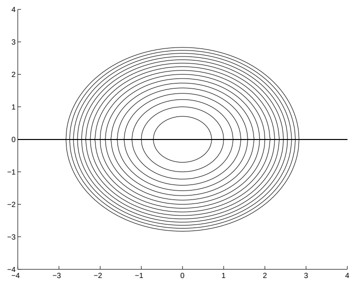
u_r, u_φ and u_z are the $\vec{u}$ components in the directions $\vec{e}_r$, $\vec{e}_\varphi$ and $\vec{e}_z$. Here only radial component important, dependent on $\frac{m}{r}$



(c) Cylindrical polar coordinates:

$$\begin{pmatrix} u_r \\ u_\varphi \\ u_z \end{pmatrix} = \begin{pmatrix} 0 \\ k/r \\ 0 \end{pmatrix}$$

Here only tangential component important, dependent on $\frac{k}{r}$



2. Vorticity and divergence

Vorticity:

$$\vec{\omega} = \text{curl}\vec{v} = \nabla \wedge \vec{v} = \nabla \wedge \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z} \\ \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x} \\ \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \end{pmatrix}$$

2-dimensional: $\frac{\partial}{\partial z} = 0$, $w = 0$

Cartesian coordinates:	ζ	$=$	$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$
Cylindrical polar coordinates:	ζ	$=$	$\frac{1}{r}\left(\frac{\partial}{\partial r}(rv) - \frac{\partial u}{\partial \varphi}\right)$

Divergence:

$$D = \nabla \cdot \vec{v} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$$

2-dimensional: $\frac{\partial}{\partial z} = 0$, $w = 0$

Cartesian coordinates:	D	$=$	$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$
Cylindrical polar coordinates:	D	$=$	$\frac{1}{r}\left(\frac{\partial}{\partial r}(ru) + \frac{\partial v}{\partial \varphi}\right)$

(a) Vorticity

$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$$

Divergence:

$$D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -a + a = 0$$

(b) Vorticity ($r \neq 0$):

$$\zeta = \frac{1}{r}\left(\frac{\partial}{\partial r}(ru_\varphi) - \frac{\partial u_r}{\partial \varphi}\right) = 0$$

Vorticity ($r = 0$):
Gauss theorem:

$$\begin{aligned} \int_V \operatorname{div} \vec{u}\, dV &= \int_{\partial V} \vec{u} \cdot \vec{n}\, dS \\ &= \frac{m}{r} 4\pi R^2 = 4\pi R m \neq 0 \end{aligned}$$

Divergence ($r \neq 0$):

$$D = \frac{1}{r}\left(\frac{\partial}{\partial r}(ru_r) + \frac{\partial u_\varphi}{\partial \varphi}\right) = 0$$

Divergence ($r = 0$): From Gauss theorem:

$$\operatorname{div} \vec{u}|_{r=0} \neq 0$$

(c) Vorticity ($r \neq 0$):

$$\zeta = \frac{1}{r}\frac{\partial}{\partial r}(k) = 0$$

Vorticity ($r = 0$):

Divergence ($r \neq 0$):

$$D = \frac{1}{r}\frac{\partial}{\partial r}(k) = 0$$

3. Divergence, deformation and vorticity

(a) Vorticity

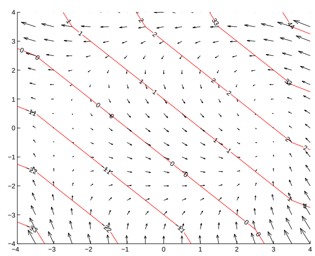
$$\zeta = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 2(x + y + a)$$

Lines with constant vorticity: $C = 2(x + y + a)$

$$y = \frac{C}{2} - x - a$$

Line between positive and negative vorticity: ($C = 0$)

$$y = -x - a$$



Flow field with isolines of vorticity

(b) Divergence

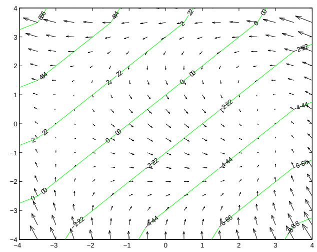
$$D = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = -2\,(x - y + a)$$

Lines with constant divergence: $C = 2\,(x + y + a)$

$$y = \frac{C}{2} + x + a$$

Line between positive and negative divergence: ($C = 0$)

$$y = x + a$$



Flow field with isolines of divergence

(c) Deformation

$$\begin{aligned} \text{Def} &= \begin{pmatrix} \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{pmatrix} = \begin{pmatrix} -2\,(-x - y + a) \\ 2\,(x - y - a) \end{pmatrix} \\ (\text{Def})^2 &= 4\left((-x - y + a)^2 + (x - y - a)^2\right) \\ &= 8\left((x - a)^2 + y^2\right) = R^2 \end{aligned}$$

Circle with centre in $(a, 0)$

4. Divergence

Continuity equation:

$$\begin{aligned} \frac{D\rho}{Dt} + \rho\vec{\nabla}\cdot\ \vec{v} &= 0 \\ \Rightarrow\ \vec{\nabla}\cdot\vec{v} = \text{div}\vec{v} &= -\frac{1}{\rho}\frac{D\rho}{Dt} \\ &\cong \frac{\Delta\rho}{\rho}\frac{1}{\Delta t} \\ &\cong \pm 0.1 \cdot \frac{1}{7200\text{s}} \\ &\cong \pm 1.4 \cdot 10^{-5}\text{s}^{-1} \end{aligned}$$

- $\Delta\rho$ positive (compression):
  $\Rightarrow$ div ρ negative (convergence)

- $\Delta\rho$ negative (dilatation):
  $\Rightarrow$ div ρ positive (divergence)