UWIS, Mathematik 1, Lösung Serie 12

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1 DGL mit Taylorpolynomen lösen

1.1

1.2

2

2.1 exakte Werte

$$
\begin{aligned}
\cos 75^\circ + i\sin 75^\circ =& e^{i75^\circ} = e^{i\left(\frac{\pi}{4}\frac{\pi}{6}\right)} = e^{i\left(\frac{\pi}{4}\right)}e^{i\left(\frac{\pi}{6}\right)}\\
=& \left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)\\
=& \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)\\
=& \frac{\sqrt{3}}{2\sqrt{2}} + i\frac{1}{2\sqrt{2}} + i\frac{\sqrt{3}}{2\sqrt{2}} - \frac{1}{2\sqrt{2}} = \frac{\sqrt{3}-1}{2\sqrt{2}} + i\frac{\sqrt{3}+1}{2\sqrt{2}}
\end{aligned}
$$

Somit ist $\sin 75^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$ und $\cos 75^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}$.

$$
\begin{aligned}
\cos 105^\circ + i\sin 105^\circ =& e^{i(75^\circ+30^\circ)} = \left(\frac{\sqrt{3}-1}{2\sqrt{2}} + i\frac{\sqrt{3}+1}{2\sqrt{2}}\right)e^{i\frac{\pi}{6}}\\
=& \left(\frac{\sqrt{3}-1}{2\sqrt{2}} + i\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)\\
=& \frac{\left(\sqrt{3}-1\right)\sqrt{3}}{4\sqrt{2}} + i\frac{\sqrt{3}-1}{4\sqrt{2}} + i\frac{\left(\sqrt{3}+1\right)\sqrt{3}}{4\sqrt{2}} - \frac{\sqrt{3}+1}{4\sqrt{2}}\\
=& \frac{3-\sqrt{3}-1-\sqrt{3}}{4\sqrt{2}} + i\frac{\sqrt{3}-1+\sqrt{3}+3}{4\sqrt{2}}\\
=& \frac{2-2\sqrt{3}}{4\sqrt{2}} + i\frac{2\sqrt{3}-2}{4\sqrt{2}} = \frac{1-\sqrt{3}}{2\sqrt{2}} + i\frac{\sqrt{3}+1}{2\sqrt{2}}
\end{aligned}
$$

Somit ist $\sin 105^\circ = \frac{\sqrt{3}+1}{2\sqrt{2}}$ und $\cos 105^\circ = \frac{1-\sqrt{3}}{2\sqrt{2}}$.

2.2 Funktionen von 3α

$$\begin{aligned}
e^{i3\alpha} =& e^{i\alpha}e^{i\alpha}e^{i\alpha} = (\cos\alpha + i\sin\alpha)(\cos\alpha + i\sin\alpha)(\cos\alpha + i\sin\alpha)\\
=&(\cos^2\alpha + 2i\sin\alpha\cos\alpha + \sin^2\alpha)(\cos\alpha + i\sin\alpha)\\
=&\cos^3\alpha + 2i\sin\alpha\cos^2\alpha - \sin^2\alpha\cos\alpha + i\sin\alpha\cos^2\alpha\\
&+ 2i^2\sin^2\alpha\cos\alpha - i\sin^3\alpha\\
=&\cos^3\alpha - 3\sin^2\alpha\cos\alpha + i(3\sin\alpha\cos^2\alpha - \sin^3\alpha)\\
=&\cos^3\alpha - 3(1-\cos^2\alpha)\cos\alpha + i(3\sin\alpha(1-\sin^2\alpha) - \sin^3\alpha)\\
=&4\cos^3\alpha - 3\cos\alpha + i(3\sin\alpha - 4\sin^3\alpha)
\end{aligned}$$

$$\begin{aligned}
\sin 3\alpha &= 3\sin\alpha - 4\sin^3\alpha\\
\cos 3\alpha &= 4\cos^3\alpha - 3\cos\alpha
\end{aligned}$$

2.3 Linearfaktorzerlegung von $x^4 - 2x^2 - 8$

Substitution mit $y = x^2 \Rightarrow x^4 - 2x^2 - 8 = y^2 - 2y - 8 = (y-2)(y+4)$. Durch Rücksubstitution ergibt sich:

$$\begin{aligned}
\sqrt{y_1} &= \pm\sqrt{2}\\
\sqrt{y_2} &= \pm i2
\end{aligned}$$

Somit ist die Linearfaktorzerlegung $(x+\sqrt{2})(x-\sqrt{2})(x+i2)(x-i2)$.

2.4 Polynom mit gegebenen Nullstellen

$$\begin{aligned}
(x-1)(x-(1+i))(x-(1-i)) &= (x-1)(x-1-i)(x-1+i)\\
&= (x-1)(x^2 - x + ix - x + 1 - i - ix + i + 1)\\
= (x-1)(x^2-2x+2) &= x^3 - 2x^2 + 2x - x^2 + 2x - 2\\
&= x^3 - 3x^2 + 4x - 2
\end{aligned}$$

3 Differentialgleichung