UWIS, Mathematik 1, Lösung Serie 11

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1 Bestimme Re z, Im z, $|z|$, $\arg z$

1.1

$$|\frac{1}{z}| = 3 \quad \Rightarrow \quad |z| = \frac{1}{3}$$
$$\arg \bar{z} = \frac{4\pi}{3} \quad \Rightarrow \quad \arg z = \frac{2\pi}{3}$$

$$\begin{aligned}
\frac{a}{\frac{1}{3}} &= \cos\frac{2\pi}{3} \\
3a &= -\frac{1}{2} \\
a &= -\frac{1}{6} \\
\sqrt{a^2+b^2} &= \frac{1}{3} \\
\left(-\frac{1}{6}\right)^2 + b^2 = \frac{1}{36} + b^2 &= \frac{1}{9} \\
b^2 &= \frac{1}{9} - \frac{1}{36} = \frac{4-1}{36} \\
b &= \sqrt{\frac{3}{36}} = \sqrt{\frac{1}{12}} \\
\text{Re } z &= -\frac{1}{6} \\
\text{Im } z &= \frac{1}{\sqrt{12}}
\end{aligned}$$

1.2

$$
\begin{aligned}
z &= \frac{-1+i\sqrt{3}}{1+i\sqrt{3}} = \frac{(-1+i\sqrt{3})(1-i\sqrt{3})}{(1+i\sqrt{3})(1-i\sqrt{3})} = \frac{-1+2i\sqrt{3}-3i^2}{1-3i^2} \\
z &= \frac{2+2i\sqrt{3}}{4} = \frac{1}{2} + \frac{\sqrt{3}}{2} i \\
\text{Re } z &= \frac{1}{2} \\
\text{Im } z &= \frac{\sqrt{3}}{2}
\end{aligned}
$$

$$
\begin{aligned}
|z| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} &= 1 \\
\arg z = \arctan \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \arctan \sqrt{3} &= \frac{\pi}{3}
\end{aligned}
$$

1.3

$$
\begin{aligned}
\text{Re } (i\bar{z}) = \text{Re } \ (i(a-ib)) = \text{Re } (ai - bi^2) = \text{Re } ai + b &= 3 \\
b &= 3 \\
|i\bar{z}| = |i(a-ib)| = |ia+b| = \sqrt{a^2+b^2} = \sqrt{a^2+3^2} &= 5 \\
a^2 + 9 &= 25 \\
a^2 &= 16 \\
a &= \pm 4 \\
\text{Re } z &= \pm 4 \\
\text{Im } z &= 3 \\
|z| &= 5 \\
\arg z &= \pm \arctan \frac{3}{4}
\end{aligned}
$$

1.4

$$
\begin{aligned}
2 &= \operatorname{Im}\frac{z}{i} = \operatorname{Im}\frac{a+ib}{i} = \operatorname{Im}\frac{(a+ib)i}{i^2} = \operatorname{Im}\frac{ia-b}{-1} = \operatorname{Im}\,(b-ia)\\
a &= -2\\
-\frac{\pi}{3} &= \arg(i\bar{z}) = \arg i(a+ib) = \arg(ia-b) = \arg(-2i-b)\\
\frac{\pi}{3} &= \arg(-b+2i)\\
\frac{2}{-b} &= \tan\frac{\pi}{3} = \sqrt{3}\\
b &= -\frac{2}{\sqrt{3}}\\
\operatorname{Re} z &= -2\\
\operatorname{Im} z &= -\frac{2}{\sqrt{3}}\\
|z| &= \sqrt{(-2)^2 + \left(-\frac{2}{\sqrt{3}}\right)^2} = \frac{4}{\sqrt{3}}\\
\arg z &= \arctan\frac{-\frac{2}{\sqrt{3}}}{-2} = \arctan\frac{1}{\sqrt{3}} = \frac{\pi}{6}
\end{aligned}
$$

2 Schreibe in der Form $z = a + ib$

2.1

2.1.1 z_1

$$
\begin{aligned}
\frac{\sqrt{3}+1-i\left(\sqrt{3}+1\right)}{1+i} &= \frac{\left(\sqrt{3}+1-i\sqrt{3}-i\right)(1-i)}{(1+i)(1-i)}\\
&= \frac{\sqrt{3}+1-i\sqrt{3}-i-i\sqrt{3}-i-i^2\sqrt{3}-i^2}{1-i^2} = \frac{2\sqrt{3}+2-2\sqrt{3}i-2i}{2}\\
&= \sqrt{3}+1+i(-\sqrt{3}-1)
\end{aligned}
$$

2.1.2 z_2

$$
\begin{aligned}
\left(\sqrt{3}+i\right)^6(1-i) &= \left(\sqrt{\sqrt{3}^2+1^2}\quad e^{\arctan\left(\frac{1}{\sqrt{3}}\right)}\right)^6(1-i)\\
&= 2^6\underbrace{e^{\frac{\pi}{6}6}}_{=-1}(1-i) = -64(1-i) = -64+i64
\end{aligned}
$$

2.1.3 z_3

$e^{-i\frac{\pi}{4}} \Rightarrow$ gleichschenkliges ($a = b$), rechtwinkliges Dreieck wobei die Hypotenuse die Länge 1 hat.

$$
\begin{aligned}
1^2 = a^2 + a^2 \\
\sqrt{\frac{1}{2}} &= a \\
a + ib &= \frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}
\end{aligned}
$$

2.1.4 z_4

$$
\begin{aligned}
\frac{1}{e^{i\frac{2\pi}{3}}} &= e^{-i\frac{2\pi}{3}} = e^{i\frac{4\pi}{3}} \\
\text{Im } z &= sin\frac{4\pi}{3} = -\frac{\sqrt{3}}{2} \\
\text{Re } z &= cos\frac{4\pi}{3} = -\frac{1}{2} \\
a + ib &= -\frac{\sqrt{3}}{2} - i\frac{1}{2}
\end{aligned}
$$

2.1.5 z_5

$$
\begin{aligned}
i\,\underbrace{e^{-i\frac{\pi}{6}}}_{z_1} \\
\text{Im } z_* &= sin-\frac{\pi}{6} = -\frac{1}{2} \\
\text{Re } z_* &= cos-\frac{\pi}{6} = \frac{\sqrt{3}}{2} \\
i\left(\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) &= i\frac{\sqrt{3}}{2} + \frac{1}{2} = \frac{1}{2} + i\frac{\sqrt{3}}{2}
\end{aligned}
$$

2.1.6 z_6

$$
i^i = \left(e^{i\frac{\pi}{2}}\right)^i = e^{i^2\frac{\pi}{2}} = e^{-\frac{\pi}{2}} \approx 0.209
$$

2.2

2.2.1

$$
\arg z_1 = \arctan\left(\frac{-\sqrt{3} - 1}{\sqrt{3} + 1}\right) = \arctan -1 = \frac{\pi}{4}
$$

2.2.2

$$z_3{}^2 z_4 = \left(\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}\right)^2 \left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = \left(\frac{1}{2} - 2i\frac{1}{2} + i^2\frac{1}{2}\right)\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right)$$
$$= -i\left(-\frac{\sqrt{3}}{2} - i\frac{1}{2}\right) = i\frac{\sqrt{3}}{2} + i^2\frac{1}{2} = \frac{1}{2} + i\frac{\sqrt{3}}{2}$$

2.2.3

$$\frac{z_4}{z_5} = \frac{-\frac{\sqrt{3}}{2} - i\frac{1}{2}}{\frac{1}{2} + i\frac{\sqrt{3}}{2}} = -\frac{\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)}{\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)}$$
$$= \frac{\frac{\sqrt{3}}{4} - i\frac{3}{4} + i\frac{1}{4} + \frac{\sqrt{3}}{4}}{\frac{1}{4} + \frac{3}{4}} = \frac{2\sqrt{3}}{4} - i\frac{2}{4} = \frac{\sqrt{3}}{2} - i\frac{1}{2}$$

3 Bestimme alle komplexen Zahlen z

3.1

Polynom zweiten Grades ⇒ 2 zwei Lösungen.

$$\begin{aligned}
z^2 = (a+ib)^2 &= i \\
a^2 + 2iab - b^2 &= i \\
a^2 - b^2 &= 0 \Rightarrow a = \pm b \\
2iab &= i \\
ab &= \frac{1}{2} \text{ da } a = \pm b \Rightarrow a = b \\
a^2 &= \frac{1}{2} \\
a = b &= \pm\frac{1}{\sqrt{2}}
\end{aligned}$$

Lösungen: $\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ und $-\frac{1}{\sqrt{2}} - i\frac{1}{\sqrt{2}}$

3.2

Eine Lösung von $z^3 = -1$ kann erraten werden: $(-1)^3 = -1$.Die anderen drei Lösungen sind gleichmässig auf dem Einheitskreis verteilt da $-1 = e^{i\pi}$ ist sind die zwei Lösungen: $e^{i\frac{\pi}{3}}$ und $e^{i\frac{2\pi}{3}}$.

$$\begin{aligned}
z_1 &= -1 \\
z_2 &= \cos\frac{\pi}{3} + i\sin\frac{\pi}{3} = \frac{1}{2} + i\frac{\sqrt{3}}{2} \\
z_3 &= \frac{1}{2} - i\frac{\sqrt{3}}{2}
\end{aligned}$$

3.3

$$\begin{aligned}
z^4 =& \frac{81\left(\sqrt{3}+i\right)}{-\sqrt{3}+i} = \frac{81\left(\sqrt{3}+i\right)\left(-\sqrt{3}-i\right)}{\left(-\sqrt{3}+i\right)\left(-\sqrt{3}-i\right)} = \frac{81\left(-3-2i\sqrt{3}+1\right)}{3+1} \\
=& \frac{81\cdot -2\left(1+i\sqrt{3}\right)}{4} = -\frac{81}{2} - \frac{i81\sqrt{3}}{2}
\end{aligned}$$

$$\begin{aligned}
r &= \sqrt{\frac{81^2}{4}+\frac{3\cdot 81^2}{4}} = \sqrt{\frac{81^2(1+3)}{4}} = 81 \\
\varphi &= \arctan\frac{-\frac{81\sqrt{3}}{2}}{-\frac{81}{2}} = \arctan\sqrt{3} = \frac{\pi}{3}
\end{aligned}$$

$$\begin{aligned}
z^4 &= 81e^{i\left(\frac{\pi}{3}+2k\pi\right)} \\
z &= 3e^{i\left(\frac{\pi}{3\cdot 4}+\frac{2k\pi}{4}\right)} = 3e^{i\left(\frac{\pi}{12}+\frac{6k\pi}{12}\right)} \\
z_1 &= 3e^{i\frac{\pi}{12}} \\
z_2 &= 3e^{i\frac{7\pi}{12}} \\
z_3 &= 3e^{i\frac{13\pi}{12}} \\
z_4 &= 3e^{i\frac{19\pi}{12}}
\end{aligned}$$

3.4

$$\begin{aligned}
4z\bar{z}+(z-\bar{z})^2 &= 3 \\
4(a+ib)(a-ib)+((a+ib)-(a-ib))^2 &= 3 \\
4(a^2+b^2)(2ib)^2 &= 3 \\
4a^2+4b^2+4i^2b^2 &= 3 \\
4a^2+4b^2-4b^2 &= 3 \\
4a^2 &= 3 \\
a &= \pm\frac{\sqrt{3}}{2} \\
b &= \in\mathbb{R}
\end{aligned}$$

3.5

$$\begin{aligned}
iz^2+2z+\sqrt{3} = 0 \\
z_1 = \frac{-2+\sqrt{2^2-4i\sqrt{3}}}{2i} \\
z_2 = \frac{-2-\sqrt{2^2-4i\sqrt{3}}}{2i}
\end{aligned}$$

4 Teich mit Forellen

4.1

$$\begin{aligned}
V_0 &= 17280\text{lzur Zeit } t_0 \\
N(t) &= \text{Anzahl Forellen im Teich zur Zeit } t_0 \\
N(0) &= 1864 \text{ Forellen zur Zeit } t_0 \\
z &= 12\frac{\text{l}}{\text{min}} \\
a &= 8\frac{\text{l}}{\text{min}} \\
z_F &= \frac{2}{40}\frac{\text{Forellen}}{\text{l}} \\
k &= 12\frac{2}{40} = \frac{3}{5}\frac{\text{Forellen}}{\text{min}} \text{ Zufluss Forellen} \\
\alpha &= \text{Proportionalitätsfaktor}
\end{aligned}$$

Wassermenge im Teich ist egal, da die Anzahl abfliessenden Forellen proportinal zur Anzahl Forellen im Teich ist.

$$\begin{aligned}
N(0+\Delta t) &= N(0) + \Delta t k - \Delta t N(0)\alpha \\
\frac{N(0+\Delta t) - N(0)}{\Delta t} &= k - N(0)\alpha \\
\dot{N}(t) &= k - N(t)\alpha
\end{aligned}$$

$$\begin{aligned}
N(t) &= ce^{-\alpha t} + k \\
N(0) = 1864 &= c\underbrace{e^{-\alpha 0}}_{=1} + \frac{3}{5} \\
1864 &= c + \frac{3}{5} \\
1864 - \frac{3}{5} &= c \\
c &= \frac{9317}{5} = 1863.4 \\
N(t) &= 1863.4e^{-\alpha t} + \frac{3}{5}
\end{aligned}$$

4.2