UWIS, Mathematik 1, Lösung Serie 10

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1 Differentialgleichungen

1.1

$$\begin{aligned}
x\frac{d}{dx}(1+t) &= x^2 \\
(1+t)dx &= x^2 dt \\
\int 1+t\,dx &= \int x^2\,dt \\
-\frac{1}{x}+c_1 &= \ln(1+t)+c_2 \quad c=c_2-c_1 \\
x &= \frac{-1}{\ln(1+t)+c}
\end{aligned}$$

1.2

$$\dot{x} = -2x+e^{3t}$$

homogene Lösung:

$$\begin{aligned}
\dot{x}_h &= -2x_h \\
x_h &= c_1e^{-2t}
\end{aligned}$$

Ansatz $C(t)c_1e^{-2t}$ (auch $x=c_1e^{-2t}+Ae^{3t}$ möglich):

$$\begin{aligned}
\dot{C}(t)c_1e^{-2t}+C(t)c_1(-2)e^{-2t} &= -2C(t)c_1e^{-2t}+e^{3t} \\
\dot{C}(t)c_1e^{-2t} &= e^{3t} \\
\dot{C}(t) &= \frac{e^{3t}}{c_1e^{-2t}} \quad c_2=\frac{1}{c_1} \\
\dot{C}(t) &= c_2e^{3t}e^{2t} \\
\dot{C}(t) &= c_2e^{5t} \\
C(t) &= \frac{c_2}{5}e^{5t}
\end{aligned}$$

Lösung:

$$\begin{aligned} x &= \frac{c_2}{5}e^{5t}c_1e^{-2t} \quad c = c_1c_2 \\ x &= \frac{c}{5}e^{3t} \end{aligned}$$

1.3

$$\begin{aligned} \dot{x} + x\sin(t) &= \sin(t) \\ \dot{x} &= -x\sin(t) + \sin(t) \\ \dot{x} &= \sin(t)(1-x) \\ \frac{\dot{x}}{(1-x)} &= \sin(t) \\ \frac{dx}{(1-x)dt} &= \sin(t) \\ \frac{1}{(1-x)}dx &= \sin(t)dt \\ \int \frac{1}{(1-x)}\,dx &= \int \sin(t)\,dt \\ \ln(1-t) &= -\cos(t) + c_1 \\ 1-x &= e^{-\cos(t)+c_1} \\ x &= 1 - e^{-\cos(t)+c_1} \\ x &= 1 - ce^{-\cos(t)} \end{aligned}$$

2 Newtonverfahren

Die Tangente der Funktion f an der Stelle x_0 (nullte Näherung der Nullstelle der Funktion f) wird mit der x-Achse geschnitten, dieser Schnittpunkt ergibt den neuen Punkt x_1 (erste Näherung der Nullstelle der Funktion f).

$$\begin{aligned} f(x_n) + f'(x_n)(x_{n+1} - x_n) &= 0 \\ f(x_n) + f'(x_n)x_{n+1} - f'(x_n)x_n &= 0 \\ f(x_n) - f'(x_n)x_n &= -f'(x_n)x_{n+1} \\ \frac{f'(x_n)x_n - f(x_n)}{f'(x_n)} &= x_{n+1} \end{aligned}$$

$$\begin{aligned} f(x) &= \tan x - x \\ f'(x) &= 1 + tan^2x - 1 = tan^2x \end{aligned}$$

$$\begin{aligned}
\frac{\left(tan^2 x_n\right) x_n - (\tan x_n - x_n)}{tan^2 x} &= x_{n+1} \\
\frac{\left(tan^2 4.5\right) 4.5 - (\tan 4.5 - 4.5)}{tan^2 4.5} &= 4.4936 \\
\frac{\left(tan^2 4.49\right) 4.49 - (\tan 4.49 - 4.49)}{tan^2 4.49} &= 4.4934
\end{aligned}$$

3 Mac Laurinsche Reihen

3.1

3.1.1

$$\begin{aligned}
f(t) &= (1+t)^{\frac{1}{2}} \\
f'(t) &= \frac{1}{2}(1+t)^{-\frac{1}{2}} \\
f''(t) &= -\frac{1}{4}(1+t)^{-\frac{3}{2}} \\
f'''(t) &= \frac{3}{8}(1+t)^{-\frac{5}{2}} \\
f^{(4)}(t) &= -\frac{15}{16}(1+t)^{-\frac{7}{2}}
\end{aligned}$$

$$\begin{aligned}
T(t) =& (1+0)^{\frac{1}{2}} + \frac{1}{2}(1+0)^{-\frac{1}{2}}(t-0) - \frac{1}{4}(1+0)^{-\frac{3}{2}}(t-0)^2\frac{1}{2} \\
&+ \frac{3}{8}(1+0)^{-\frac{5}{2}}(t-0)^3\frac{1}{6} - \frac{15}{16}(1+0)^{-\frac{7}{2}}(t-0)^4\frac{1}{24} \\
T(t) =& 1 + \frac{1}{2}t - \frac{1}{8}t^2 + \frac{1}{16}t^3 - \frac{5}{128}t^4
\end{aligned}$$

3.1.2

eine Ableitung mehr als in 3.1.1 und $g^{(n)}(t) = f^{(n+1)}(t)$

$$\begin{aligned}
g(t) &= (1+t)^{-\frac{1}{2}} \\
g'(t) &= -\frac{1}{2}(1+t)^{-\frac{3}{2}} \\
g''(t) &= \frac{3}{4}(1+t)^{-\frac{5}{2}} \\
g'''(t) &= -\frac{15}{8}(1+t)^{-\frac{7}{2}} \\
g^4(t) &= \frac{105}{16}(1+t)^{-\frac{9}{2}}
\end{aligned}$$

$$T(t) = 1 - \frac{1}{2}t + \frac{3}{8}t^2 - \frac{5}{16}t^3 + \frac{35}{128}t^4$$

3.1.3

$$\begin{aligned}
f(t) &= \ln\left(\frac{1+t}{1-t}\right) = \ln(1+t) - \ln(1-t) \\
f'(t) &= \frac{1}{1+t} + \frac{1}{1-t} \\
f''(t) &= -\frac{1}{(1+t)^2} + \frac{1}{(1-t)^2} \\
f'''(t) &= \frac{2}{(1+t)^3} + \frac{2}{(1-t)^3} \\
f^{(4)}(t) &= -\frac{6}{(1+t)^4} + \frac{6}{(1-t)^4} \\
f^{(5)}(t) &= \frac{24}{(1+t)^5} + \frac{24}{(1-t)^5}
\end{aligned}$$

$$T(t) = 2t + \frac{2}{3}t^3 + \frac{2}{5}t^5$$

3.1.4

$$\begin{aligned}
f(t) &= \tan(t) \\
f'(t) &= 1 + \tan^2(t) \\
f''(t) &= 2\tan(t)\left(1+\tan^2(t)\right) = 2\tan(t) + 2\tan^3(t) \\
f'''(t) &= 2 + 2\tan^2(t) + 6\tan^2(t)\left(1+\tan^2(t)\right) \\
&= 2 + 8\tan^2(t) + 6\tan^4(t) \\
f^{(4)}(t) &= 16\tan(t)\left(1+\tan^2(t)\right) + 24\tan^3(t)\left(1+\tan^2(t)\right) \\
&= 16\tan(t) + 40\tan^3(t) + 24\tan^5(t) \\
f^{(5)}(t) &= 16\left(1+\tan^2(t)\right) + 120\tan^2(t)\left(1+\tan^2(t)\right) + \\
&\quad 120\tan^4(t)\left(1+\tan^2(t)\right) \\
&= 16 + 136\tan^2(t) + 240\tan^4(t) + 120\tan^6(t)
\end{aligned}$$

3.2

$$\begin{aligned}
\frac{1+t}{1-t} &= 2 \\
1+t &= 2-2t \\
3t &= 1 \\
t &= \frac{1}{3}
\end{aligned}$$

$$
\begin{aligned}
T\left(\frac{1}{3}\right) &= 2\frac{1}{3} + \frac{2}{3}\left(\frac{1}{3}\right)^3 + \frac{2}{5}\left(\frac{1}{3}\right)^5 = \frac{2}{3} + \frac{2}{81} + 21215 \\
T\left(\frac{1}{3}\right) &= \frac{842}{1215} = 0.69300
\end{aligned}
$$

Zum Vergleich $\ln(2) = 0.69315$ der Fehler ist sicher kleiner $t^7 < 0.00046$.

4 Taylorpolynome verschiedenen Grades

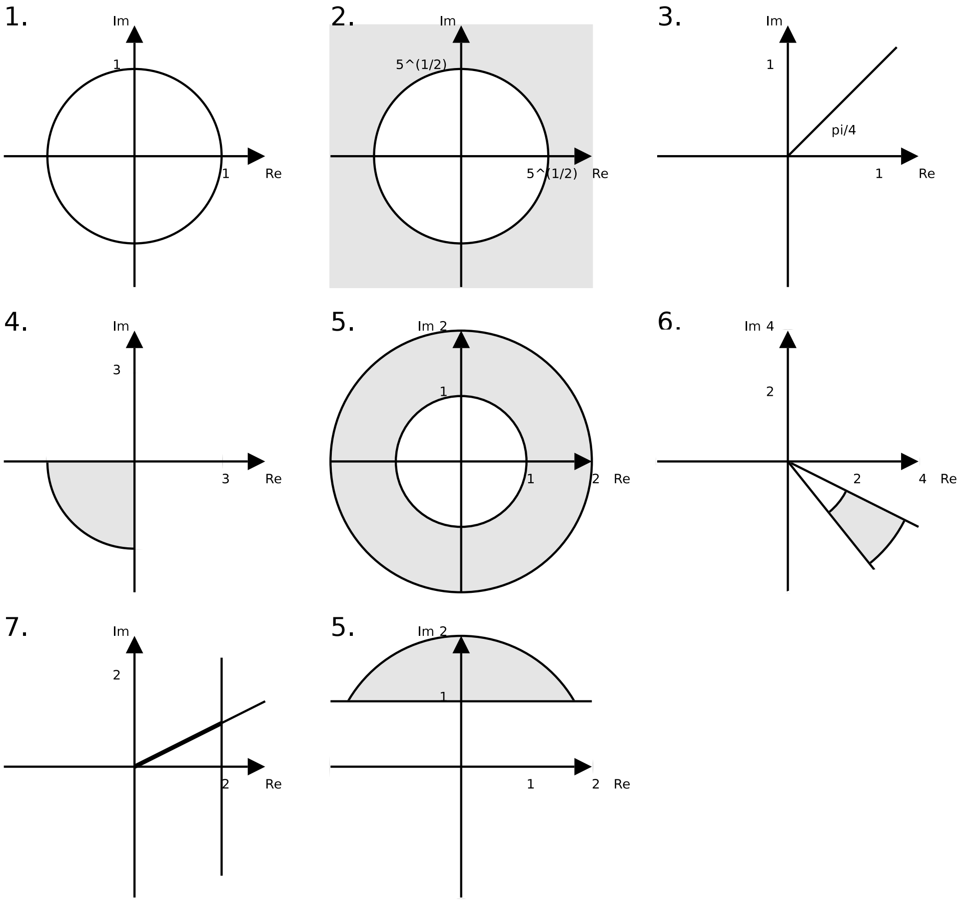
```
> series(sin(t),t=0,1);
                           O(t)

> series(sin(t),t=0,3);
                                 3
                         t + O(t )

> series(sin(t),t=0,5);
                             3       5
                    t - 1/6 t  + O(t )

> series(sin(t),t=0,7);
                        3            5       7
               t - 1/6 t  + 1/120 t  + O(t )
```

allgemein:

$$\sin t = \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k+1}}{(2k+1)!}$$

5 komplexe Zahlen

5.1



5.2

5.2.1

$$\begin{aligned} z &= \frac{5-5i}{-1+2i} = \frac{(5-5i)(-1-2i)}{(-1+2i)(-1-2i)} = \frac{-5-10i+5i+10i^2}{1-4i^2} = \frac{-15-5i}{5} \\ z &= -3-i \end{aligned}$$

5.2.2

$$\begin{aligned} z &= \frac{1}{(1+2i)^3} = \frac{(1-2i)^3}{((1+2i)(1-2i))^3} = \frac{(-3-4i)(1-2i)}{(1-4i^2)^3} = \frac{-11+2i}{5^3} \\ z &= \frac{-11}{125} + \frac{2}{125}i \end{aligned}$$